Classical Signature Change in the Black Hole Topology

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Abstract

Investigations of classical signature change have generally envisaged applications to cosmological models, usually a Friedmann-Lemaître-Robertson-Walker model. The purpose has been to avoid the inevitable singularity of models with purely Lorentzian signature, replacing the neighbourhood of the big bang with an initial, singularity free region of Euclidean signture, and a signature change. We here show that signature change can also avoid the singularity of gravitational collapse. We investigate the process of re-birth of Schwarzschild type black holes, modelling it as a double signature change, joining two universes of Lorentzian signature through a Euclidean region which provides a 'bounce'. We show that this process is viable both with and without matter present, but realistic models — which have the signature change surfaces hidden inside the horizons — require non-zero density. In fact the most realistic models are those that start as a finite cloud of collapsing matter, surrounded by vacuum. We consider how geodesics may be matched across a signature change surface, and conclude that the particle 'masses' must jump in value. This scenario may be relevant to Smolin's recent proposal that a form of natural selection operates on the level of universes, which favours the type of universe we live in.

Short Title: Signature Change in Black Holes

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1. Introduction

Space-time in general relativity is usually considered to possess a metric of Lorentzian signature. Positive definite metrics, with a Euclidean signature, have come into prominence lately through the Hartle and Hawking program concerning the wave function of the universe [1] — [5]. A general aim of that program is to try get a handle on the boundary conditions of the universe, and an intriguing suggestion made in [6, 1] is that the universe has no boundary, i.e. no origin where initial conditions have to be set, which is only possible if space-time emerged from a Euclidean region preceeding a change of signature. Another interesting development is the introduction of Euclidean wormholes. These wormholes can arise in one universe and connect it either to itself or to another universe. In order to attribute a transition probability, for example, between two Lorentzian regions, integration of the action along the tube connecting the two regions under study is required. In normal Lorentzian space the path integral approach leads to oscillating behavior, and hence to non-convergence of the integral. To obtain convergence, the substitution $t \rightarrow it$ is applied, thus introducing a Euclidean signature. This means in effect that we have two Lorentzian regions connected through a classically forbidden Euclidean region.

Paralleling the Quantum cosmology program, papers [7] — [10] pointed out that the possibility of a change in the signature of the metric is not restricted to a quantum description of General Relativity. It was shown in [8, 9] that classical General Relativity does not prevent the existence of Euclidean regions and some examples of signature change in the Friedmann-Lemaître-Robertson-Walker metric were produced. Further investigations were pursued in [11] — [30]. Even though the metric signature is invisible to the Einstein Field Equations, it should be noted [8, 19, 23] that a change of signature is not, either g_{00} goes through zero, in which case the metric is degenerate and g^{00} singular there, or both g_{00} and g^{00} jump from positive to negative values, in which case the metric is discontinuous. In either case, the Einstein Field equations cannot be defined in the usual way at the signature change [23, 27].

The Friedmann-Lemaître-Robertson-Walker model has been considered repeatedly in both classical and quantum cosmological signature change, but few other models have been considered, especially in the classical signature change literature. This paper constructs a classical model of signature change within the black hole topology, using Schwarzschild and Lemaître-Tolman models — i.e. a Kruskal-Szekeres type of manifold. It examines the transition from a black hole, through a signature change to a Euclidean region which reverses the collapse process, leading to a second signature change, and the birth of a white hole and a new universe. We also determines whether the signature change surface can be hidden inside the horizon. It continues the approach of papers [8, 9, 19] by exploring strictly classical signature changes in the Schwarzschild and Lemaître-Tolman metrics.

The investigation of transitions between Lorentzian geometries through a Euclidean region are also of interest when considered in conjunction with Smolin's idea [31]. Smolin's hypothesis is a proposed mechanism for determining the particular values of fundamental physical constants observed today, and thus justifying the anthropic principle. In Smolin's paper, life supporting characteristics are linked to the existence of stars whose abundance is linked to the abundance of black holes. It is envisaged that each 'universe' either expands and re-collapses or expands indefinitely, possibly forming one or more black holes. Instead of classical singularities occuring — a crunch singularity or black hole future singularities — quantum cosmological tunneling gives birth to new universes, and hence a 'natural history' of universes arises. Furthermore, Smolin proposes that the process of tunnelling generates small random changes — 'mutations' — in the values of the physical constants. Those combinations of values for which the universe generates many black holes will lead to large numbers of offspring having very similar values. Thus, after the passage of many generations of universes, the population of universes will come to be dominated by those that generate lots of stars and black holes. This parallels natural selection in that the 'fittest' universes reproduce prolifically, but differs in that all blood lines (sets of constants) survive.

In what follows, we consider the junction conditions at a signature change in a Schwarzschild type metric, the choice of signature change surface, the form of the metric in the Euclidean region, and how geodesics should be propagated through. Generalisations lead naturally to the Lemaître-Tolman metric and its Kantowski-Sachs limit, which allow more interesting results.

The first step one has to take is to ensure that the various regions composing the space match geometrically. We adopt the Darmois junction conditions [32], and the application of them to signature change as presented in [19].

2. Junction Conditions and Conservation Laws

Conventions

We here work with 4-dimensional manifolds of Lorentzian and Euclidean signatures $(-+++)$ and $(++++)$ respectively. Greek indices range 0-3 and Latin indices 1-3. Subscripts E and L denote quantities defined in or evaluated in Euclidean and Lorentzian regions respectively, and expressions without such subscripts are valid in either region. These may also be written as superscripts, to avoid confusion with tensor indices. Geometric units are used, $G = c = 1$, and the cosmological constant is set to zero, $\Lambda = 0$.

Darmois Matching conditions

In standard Darmois matching, where no signature change occurs, space is composed of two regions, V^+ and V^-, with a common boundary surface Σ. More precisely, an isomorphism $\psi : \Sigma^+ \to \Sigma^-$ allows us to identify the boundaries of $V^\pm$, $\Sigma^+ = \Sigma^- = \Sigma$. The two regions have coordinate charts x^μ_+ and x^μ_- and metrics $g^+_{\mu\nu}$ and $g^-_{\mu\nu}$ respectively. Setting the intrinsic coordinates of the junction surface to be $\xi^i_+ = \xi^i_- = \xi^i$, the locus of the surface is given parametrically in $V^\pm$ by $x^\mu_\pm = x^\mu_\pm(\xi^i)$, or by $\Xi^\pm(x^\mu_\pm) = 0$. We write $Q\,|^\pm$ to denote evaluation of some quantity Q in the limit as the surface is approached from either region, and $[Q]$ to denote the difference between the two limiting values

$$[Q] = Q\,|^+ \ - Q\,|^- \tag{1}$$

The Darmois conditions [32] require the continuity of the first and second fundamental forms of the junction surface — i.e. the intrinsic metric and the extrinsic curvature. The intrinsic metric is obtained by projecting the 4-metric onto Σ using the basis vectors e^μ_i of Σ

$$\overset{3}{g}_{ij} = g_{\mu\nu} e^\mu_i e^\nu_j \,, \qquad e^\mu_i = \frac{\partial x^\mu}{\partial \xi^i} \tag{2}$$

The extrinsic curvature describes the surface's shape in the enveloping space, and is the projection onto Σ, of the rate of change of the surface normal n^μ in the enveloping space, with respect to position on Σ.

$$K_{ij} = (\nabla_\mu n_\nu)\, e^\mu_i e^\nu_j = -n_\lambda \left(\frac{\partial^2 x^\lambda}{\partial \xi^i \partial \xi^j} + \Gamma^\lambda_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^i} \frac{\partial x^\nu}{\partial \xi^j} \right), \qquad n_\nu = \pm \frac{\partial_\nu \Xi}{\sqrt{\epsilon_n \partial^\mu \Xi \partial_\mu \Xi}} \tag{3}$$

where $\epsilon_n = n^\nu n_\nu = +1$ if Σ is time-like and -1 if it is space-like. To conform with (1), the sign in (3) is set so that the $n^\nu_\pm$ point from V^- to V^+ on both sides of Σ. The Darmois conditions may now be given as:

$$[\overset{3}{g}_{ij}] = 0 \quad \& \quad [K_{ij}] = 0 \tag{4}$$

In the constant signature case, according to [33], these are equivalent to the Lichnerowicz matching conditions [34], whereas the O'Brien and Synge conditions [35] are too restrictive.

When we introduce a signature change at Σ, the equivalence between the Darmois and Lichnerowicz conditions breaks down. Both the Lichnerowicz and O'Brien and Synge conditions insist that all the 4-d metric components be matched on either side of the junction surface, leading to a degenerate metric, a non-affine time coordinate, and breakdown of the Einstein field equations. We select the Darmois matching conditions as they are invariant to the coordinates chosen on either side. They require no modification at a surface of signature change. In fact they are blind to the change of signature, thus extending the signature blindness of the Einstein field equations. A signature change surface is necessarily space-like, so $\epsilon_n = n_\mu n^\mu = +1$ in the Euclidean region, and -1 in the Lorentzian region.

Conservation Laws

In [19, 20] the implications of signature change for conservation laws are worked out. Conservation laws are based on the divergence theorem — i.e. the components version of Stokes theorem for a 3-form in a metric space. The theorem requires a region W, bounded by a closed surface S with outward pointing unit normal m_α, smooth non-zero volume elements d^4W and d^3S on W and S respectively, a smooth non-zero metric, so that the inverse metric is well defined, and a smooth field Ψ^δ:

$$\oint_S \Psi^\delta m_\delta d^3S = \int_W \nabla_\delta \Psi^\delta d^4W \tag{5}$$

It should be noted that n^α is the normal to the junction surface Σ, and m_α is the normal to S, the closed boundary of W;

These conditions are not satisfied through a signature change. Thus physical conservation laws need to be revised. For the electro-magnetic field we work with the 4-current $\Psi^\delta = J^\delta$, and for the gravitational field, a component of the Einstein tensor $\Psi^\delta = G^{\gamma\delta} v_\gamma$ where v_γ is some suitable smooth vector field. Since v_γ and v^γ are not both smooth through a signature change, $\Psi^\delta = G^\delta_\gamma v^\gamma$ is also considered.

Firstly, at a boundary where no signature change occurs, the Darmois junction conditions may be used to patch together two regions that adjoin the boundary on either side, and within which the divergence theorem does hold. It is shown that these conditions, which give rise to Israel's identities [36] for the Einstein tensor,

$$\begin{aligned} [G_{\mu\nu} n^\mu n^\nu] &= 0 && (6) \\ [G_{\mu\nu} e_i^\mu n^\nu] &= 0 && (7) \end{aligned}$$

where

$$\begin{aligned} G_{\mu\nu} n^\mu n^\nu &= \frac{1}{2}\{K^2 - K_{ij}K^{ij} - \epsilon_n {}^3\!R\} && (8) \\ G_{\mu\nu} e_i^\mu n^\nu &= {}^3\nabla_j K_i^j - {}^3\nabla_i K && (9) \end{aligned}$$

${}^3\!R$ and ${}^3\nabla_i$ being the intrinsic curvature invariant and covariant derivative of the 3-surface, and $K = g^{ij} K_{ij} = K_j^j$, are sufficient to ensure conservation of energy-momentum through Σ. Combined with suitable junction conditions on the electro-magnetic field, they also ensure conservation of 4-current, with similar results applying to other fields.

At a surface of signature change, ϵ_n now flips sign across Σ, and this leads to modified Israel identities

$$\begin{aligned} [G_{\mu\nu} n^\mu n^\nu] &= -{}^3\!R && (10) \\ [G_{\mu\nu} e_i^\mu n^\nu] &= 0 && (11) \end{aligned}$$

It is necessary to distinguish two normals to Σ: $l_\delta = \partial\xi^0/\partial x^\delta = \overline{e}^0_\delta$ and $n^\delta = \partial x^\delta/\partial \xi^0 = e_0^\delta$ where $n^\gamma n_\gamma = \epsilon_n = l_\gamma l^\gamma$ and $g_{\gamma\delta} n^\delta = \epsilon_n l_\gamma$. A similar analysis of the divergence theorem through Σ is made, paying careful attention to index position, the definition of the extrinsic curvature, and the directions of the various normals. It is found that, in the process of patching together the two divergence theorems on either side of the signature change, the combined theorem aquires a surface term, so conservation laws must in general be modified. The result is

$$\oint_S \Psi^\beta m_\beta \, d^3S - \int_{S_o} E \, d^3S_o = \int_W \nabla_\beta \Psi^\beta \, d^4W \tag{12}$$

where

$$E = (\Psi^\alpha_+ l^+_\alpha - \Psi^\alpha_- l^-_\alpha) = [\Psi^\alpha l_\alpha] \tag{13}$$

and S_o is the region of Σ enclosed by S. For each of the four choices $\Psi^\delta = G^{\gamma\delta} v_\gamma$, $v_\gamma = l_\gamma$ or $\overline{e}^i_\gamma$ and $\Psi^\delta = G^\delta_\gamma v^\gamma$, $v^\gamma = n^\gamma$ or e^γ_i the surface term $E = E(v_\gamma)$ or $E(v^\gamma)$ in the conservation law is

$$\begin{aligned}
E(l_\alpha) = [G^{\alpha\beta} l_\alpha l_\beta] &= -{}^3\!R & (14)\\
E(\overline{e}^i_\alpha) = [G^{\alpha\beta} l_\alpha \overline{e}^i_\beta] &= 2({}^3\nabla_j K^{ij} - {}^3g^{ij\,3}\nabla_j K) & (15)\\
E(n^\alpha) = [G^\alpha_\beta l_\alpha n^\beta] &= K^2 - K_{ij}K^{ij} & (16)\\
E(e^\alpha_i) = [G^\alpha_\beta l_\alpha e^\beta_i] &= 2({}^3\nabla_j K^j_i - {}^3\nabla_i K) & (17)
\end{aligned}$$

The main results are expressed in a way that allows any set of junction conditions to be applied at the signature change, so this conclusion is independent of choice of junction conditions, as well as being coordinate invariant. Alternative approaches to signature change, which emphasise maximum smoothness of the metric and the matter, are able to eliminate some, but not all of the surface effects. Removal of all surface effects requires that the surface of signature change not only have zero extrinsic curvature, but also have zero (3-d) Ricci scalar, which eliminates all realistic cosmological models.

At a signature change then, the Darmois conditions still impose the same number of metric conditions as was sufficient for no signature change, and they still result in a modified set of conservation laws, albeit with surface effects. These can be understood as a consequence of the change in the character of physical laws. Whilst it is of interest to follow the maximum continuity, Lichnerowicz type approach, it is argued that physically interesting scenarios may be eliminated by it. The following models are an example.

3. The Schwarzschild Case

In the usual Schwarzschild line element [37] the signs of the metric components g_{TT} and g_{RR} interchange across $R = 2M$, thus leading to reinterpertaion of the roles of R and T. Consequently it is not clear which sign we should change to introduce the signature change, assuming that the general form of the metric is retained. We shall investigate both possibilities, so we insert two new sign factors $\epsilon_T = \pm 1$ and $\epsilon_R = \pm 1$ in the metric:

$$ds^2 = \epsilon_T \left(1 - \frac{2M}{R}\right) dT^2 + \epsilon_R \left(\frac{2M}{R} - 1\right)^{-1} dR^2 + R^2 d\Omega^2 \tag{18}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. We are interested only in transitions from the standard Schwarzschild metric to a Euclidean region, so we disregard the sign combination which gives us a second (non-vacuum) Lorentzian manifold ($\epsilon_T = +1, \epsilon_R = +1$). With both the other sign combinations ($\epsilon_T = -1, \epsilon_R = +1$ and $\epsilon_T = +1, \epsilon_R = -1$) it appears that there is a Euclidean region on one side of $R = 2M$, and a "double Lorentzian" (or "Kleinian") region (with two time-like coordinates) on the other side. It will become clear that the two Euclidean 'regions' are in fact geodesically complete manifolds.

Whilst Euclidean regions have no time, there will be a direction which is the extension of the time direction in the Lorentzian region. One may determine whether the Euclidean metric is 'static' or 'dynamic' relative to this direction.

The Einstein tensor components for this metric are [38]

$$G_{TT} = \frac{-\epsilon_T(1+\epsilon_R)(1-2M/R)}{R^2}, \quad G_{RR} = \frac{(1+\epsilon_R)}{R^2(1-2M/R)}, \quad G_{\theta\theta} = 0, \quad G_{\phi\phi} = 0 \tag{19}$$

A vacuum solution requires $\epsilon_R = -1$. Transitions requiring a change of sign in ϵ_R introduce non-vacuum solutions with strangely behaved matter, which we shall consider, since we don't really know what to expect in Euclidean signature physics. The Ricci scalar and the Kretschmann scalar are

$$R^{\mu}_{\mu} = \frac{2(1+\epsilon_R)}{R^2}, \quad k = R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma} = \frac{8[(1+\epsilon_R)R^2(1-2M/R)+6M^2]}{R^6} \tag{20}$$

where $R_{\mu\nu}$ and $R_{\mu\nu\lambda\sigma}$ are the Ricci and Riemann tensors. Regardless of signature, k is always singular at $R = 0$, and never at $R = 2M$, so the Euclidean region $(\epsilon_T = +1, \epsilon_R = -1, R \geq 2M)$ has no singularities.

We now investigate whether signature change is possible on two simple spacelike surfaces — in $R \geq 2M$ a constant T surface, and in $R < 2M$ a constant R surface. More general transition surfaces will be considered using a different metric form. The surface coordinates ξ^i may then be chosen to be identically 3 of the enveloping coordinates x^μ in V^- and V^+ — viz (T, θ, ϕ) or (R, θ, ϕ). (c.f. [39] in which the Euclidean solution for vacuum with a cosmological constant is found.)

Constant T surface

The intrinsic metric, unit normal, and extrinsic curvature of Σ are given in either region by

$$\begin{aligned} d\sigma^2 &= \frac{-\epsilon_R}{(1-2M/R)} dR^2 + R^2 d\Omega^2, & \epsilon_T \epsilon_n = 1 \quad (21) \\ n^\mu &= \frac{\delta^\mu_T}{\sqrt{\epsilon_T \epsilon_n (1-2M/R)}}, \quad n^\mu n_\mu = \epsilon_n = \epsilon_T, \quad K_{ij} = 0 & \quad (22) \end{aligned}$$

and the surface effects are all zero.

$$E(l_\alpha) = \frac{2(1+\epsilon_R)}{R^2} = 0, \quad E(\bar{e}^i_\alpha) = 0, \quad E(n^\alpha) = 0, \quad E(e^\alpha_i) = 0 \tag{23}$$

The two choices of future direction for n^μ are equivalent in a static metric. The choice of a standard Schwarzschild solution in the Lorentzian region sets the sign of ϵ_R to -1, so ϵ_T flips across Σ, and this requires that $R_L \geq 2M_L$ for a space-like surface. Although (21) is singular at $R_\pm = 2M_\pm$, all constant T surfaces on the Lorentzian side intersect that point, which is only a coordinate singularity, being the middle of the Schwarzschild wormhole at its moment of maximum expansion. Applying $[{}^3\!g_{ij}] = 0$ and requiring the angular coordinates on either side to coincide, $\theta_E = \theta_L$, $\phi_E = \phi_L$, also fixes the areal radius and mass terms to be the same, $R_E = R_L$ and $M_E = M_L$. Obviously $[K_{ij}] = 0$ imposes no further constraints. Since all $T =$ constant surfaces are equivalent (for a static metric), this result is not surprising.

This matching corresponds to vacuum both before and after the signature change, but the change surface extends to spatial infinity in both exterior regions $R_L > 2M_L$. We are really seeking a change surface that is near the singularity $R = 0$ and hidden inside the horizon. No spacelike surface is further from $R = 0$ or less hidden than that of constant T.

Since the middle of the throat at maximum expansion is a stationary point of the Killing vector $\chi^\mu = \delta^\mu_T$ on the Lorentzian side, and since all Lorentzian constant T surfaces match to all Euclidean constant T surfaces, this must be a stationary point on the Euclidean side also. This leads us to suspect that it is not possible to find two separate, non-intersecting $T =$ constant surfaces in the Euclidean region. In other words, we can't construct a Euclidean region between two separate Lorentzian regions.

Constant R surface

This is the simplest non-vacuum case. The fundamental forms and surface effects are:

$$\begin{aligned} d\sigma^2 &= \epsilon_T (1-2M/R) dT^2 + R^2 d\Omega^2, & \epsilon_R \epsilon_n = 1 \quad (24) \\ n^\mu &= -\sqrt{\epsilon_R \epsilon_n (2M/R - 1)}\, \delta^\mu_R, \quad n^\mu n_\mu = \epsilon_n = \epsilon_R & \quad (25) \\ K_{TT} &= \frac{-\epsilon_T M \sqrt{\epsilon_R \epsilon_n (2M/R-1)}}{R^2}, \end{aligned}$$

$$K_{\theta\theta} = -R\sqrt{\epsilon_R\epsilon_n(2M/R-1)}, \quad K_{\phi\phi} = \sin^2\theta\, K_{\theta\theta} \tag{26}$$

$$E(l_\alpha) = \frac{-2}{R^2}, \quad E(\overline{e}^i_\alpha) = 0, \quad E(n^\alpha) = \frac{-2\epsilon_R\epsilon_n}{R^2} = \frac{-2}{R^2}, \quad E(e^\alpha_i) = 0 \tag{27}$$

where we have chosen n^μ to point in the direction of collapse, i.e. towards R decreasing. A similar analysis gives us $\epsilon_T = -1$, $R_L < 2M_L$, and ϵ_R flips; choosing $\theta_E = \theta_L$, $\phi_E = \phi_L$ and $T_E = T_L$ with $[{}^3\!g_{ij}] = 0 \Rightarrow R_E = R_L$, and $M_E = M_L$. No further restrictions are necessary to ensure $[K_{ij}] = 0$.

This demonstrates that matching can be achieved on a surface that is entirely inside the horizon, but since the Euclidean region is not empty in this case, it still leaves open the interpretation of the energy stress tensor on the Euclidean side. Further, since R is the timelike coordinate on the Lorentzian side, R is the nominal 'time' direction on the Euclidean side too, being orthogonal to the transition surface, so this Euclidean manifold, with $R < 2M$, is 'dynamic'.

Geodesic Coverage

What do the particle paths look like in the combined space? We now investigate the behavior of radial timelike geodesics. Adding the angular components of the motion should present no problem, as $\theta_\pm$ and $\phi_\pm$ are identified at Σ. One aim is to verify that the space is geodesically complete, and in the context of this paper geodesics that end on a curvature singularity are considered to be as complete as is possible. The second aim is to see how geodesics should be continued at the transition, and whether the set of all geodesics arriving at the Lorentzian side of the transition generate all possible geodesics emerging on the Euclidean side. Three schemes for continuing geodesics are considered. Two of them attempt to match particle 4-velocities (unit normal tangent vectors), and one attempts to match 4-momenta.

The geodesic equation

$$u^\mu \nabla_\mu u^\nu = 0 \tag{28}$$

with the condition for a 'time-like' unit normal

$$u^\mu u_\mu = \epsilon_n \tag{29}$$

where $\epsilon_n = \epsilon_T$ at a constant T transition or ϵ_R at a constant R transition, leads to the acceleration

$$\ddot{R} = \frac{-\epsilon_R\epsilon_n M}{R^2}\ , \tag{30}$$

and gives the following unit tangent vectors, where the signs have all been chosen so that positive h and q values always give consistently future directed infalling tangent vectors where $T > 0$.

Lorentzian: $$u^\mu = \left(\frac{h_L}{(1-2M/R)}\ , -q_L\sqrt{h_L^2 - (1-2M/R)}\ , 0, 0\right), \quad q_L = \pm 1 \tag{31}$$

There are three types of geodesics: (a) $1 - 2M/R \leq h_L^2 < 1$: geodesics recollapsing from past to future singularities, $R = 0$, with a maximum at $R = 2M/(1-h_L^2) \geq 2M$; (b) $h_L^2 > 1$: monotonically ingoing or outgoing geodesics with finite velocity $\sqrt{h_L^2 - 1}$ at $R = \infty$, reaching $R = 0$ either in the past or future; (c) $h_L^2 = 1$: marginal monotonic geodesics with zero velocity at $R = \infty$.

Euclidean, $R \geq 2M$: $$u^\mu = \left(\frac{h_E}{(1-2M/R)}\ , -q_E\sqrt{(1-2M/R) - h_E^2}\ , 0, 0\right), \quad q_E = \pm 1 \tag{32}$$

In this case there is only one type of geodesic, descending from $R = \infty$, through a minimum at $R = 2M/(1-h_E^2) \geq 2M$, and re-expanding back out. The allowed range of h_E is then $0 \leq h_E^2 \leq (1-2M/R)$, and all geodesic paths are restricted to the region $R \geq 2M$. The only geodesic reaching $R = 2M$ is the one with $h_E = 0$ which in effect is a stationary point. This is in accord with the range of R at Σ, and confirms that the region $R \geq 2M$ is a geodesically complete manifold.

Euclidean, $R < 2M$: $$u^\mu = \left(\frac{-h_E}{2M/R-1}\ , -q_E\sqrt{(2M/R-1) - h_E^2}\ , 0, 0\right), \quad q_E = \pm 1 \tag{33}$$

These geodesics all expand from $R = 0$ and re-collapse back, and have maxima at $R = 2M/(1+h_E^2) \leq 2M$, where $h_E^2 \leq (2M/R - 1)$. Thus the region $R \leq 2M$ is also geodesically complete, but does not have the desired bouncing property. ($R = 2M$ would not actually be encountered in this region of our model, since $R_\Sigma < 2M$.)

Since geodesic tangent vectors have a unit magnitude that flips sign across a signature change, it is impossible to match all components of $u^\mu_\pm$ across Σ. We consider here three possible schemes, and summarise the resulting conditions in Table 1 below:

(i) Match u^R values, with $u^\mu u_\mu = \epsilon_n$ giving a jump in the u^T values;
(ii) Match u^T values, with $u^\mu u_\mu = \epsilon_n$ giving a jump in the u^R values;
(iii) Match all components of the 4-momentum $P^\mu = mu^\mu$, allowing P_μ and $|P^\mu P_\mu| = m^2$ to jump. (This is equivalent to matching non-normalised tangent vectors, for which the metric degeneracy at Σ is irrelevant, and standard existence and uniqueness theorems guarantee that geodesic continuation of all tangent vectors is possible.)

Continuation Condition	Signature Change Surface	
	Const. R, $R < 2M$	Const. T, $R \geq 2M$
(i) $u^R_E = u^R_L$	$q_E = q_L, \quad h_E^2 = -h_L^2 = 0$	$q_E = q_L, \quad h_E^2 + h_L^2 = 2(1 - 2M/R)$
(ii) $u^T_E = u^T_L$	$h_E = h_L$	$h_E = h_L$
(iii) $P^\mu_E = P^\mu_L$	$2h_E^2 h_L^2 = (h_L^2 - h_E^2)(2M/R - 1)$, $q_E = q_L, \quad m_E h_E = m_L h_L$	$2h_E^2 h_L^2 = (h_L^2 + h_E^2)(1 - 2M/R)$, $q_E = q_L, \quad m_E h_E = m_L h_L$

Table 1. Conditions resulting from the three continuation conditions at two types of signature change that retain a Schwarzschild metric form.

Transition surface	Const. R, $R < 2M$	Const. T, $R \geq 2M$
Allowed h_L	$h_L^2 \geq 0 \geq (1 - 2M/R)$	$h_L^2 \geq (1 - 2M/R)$
Allowed h_E	$h_E^2 \leq (2M/R - 1)$	$h_E^2 \leq (1 - 2M/R)$

Table 2. Summary of allowed ranges of geodesic energy parameters on either side of the two signature change surfaces.

When these are compared with the allowed ranges of h in each region, summarised in Table 2, we see that conditions (i) and (ii) do not allow the continuation of all possible geodesics that might arrive at either type of signature change surface, whereas (iii) continues all geodesics at both types. For example, identifying $u^R_E = u^R_L$ at a constant T surface gives us

$$h_E^2 = 2(1 - 2M/R) - h_L^2, \quad R \geq 2M, \quad h_L^2 \geq (1 - 2M/R), \quad h_E^2 \leq (1 - 2M/R) \tag{34}$$

so we can easily find a large enough value for h_L^2 to make h_E imaginary. For case (iii) we find

$$\text{Constant } T \text{ transition:} \qquad \left(\frac{m_E}{m_L}\right)^2 = \left(\frac{h_L}{h_E}\right)^2 = \frac{2h_L^2}{(1 - 2M/R)} - 1 \tag{35}$$

$$\text{Constant } R \text{ transition:} \qquad \left(\frac{m_E}{m_L}\right)^2 = \left(\frac{h_L}{h_E}\right)^2 = \frac{2h_L^2}{(2M/R - 1)} + 1 \tag{36}$$

Since $h_L^2 \geq (1 - 2M/R)$ the particle's Euclidean mass is always greater than it's Lorentzian mass $m_E \geq m_L$, as well as $h_L^2 \geq h_E^2$.

Thus it turns out that condition (29), forcing the tangent vectors to be unit vectors, is too strong an assumption, and doesn't permit all particle paths to be matched through Σ. Rather, the matching of geodesic 4-momenta is the only way of extending all particle paths through a signature change. The conclusion that the 'rest-mass' parameter of a particle has to jump, is consistent with the fact [19] that in general tensors cannot be smooth through Σ in both covariant and contravariant forms, and that the density can jump across a signature change. In Fig. 1 we summarise the properties of geodesics arriving at Σ at some particular value of R_Σ. The horizontal axis is the parameter h_L^2 and the plot covers a representative range of permissible h_L^2 values for the chosen R_Σ.

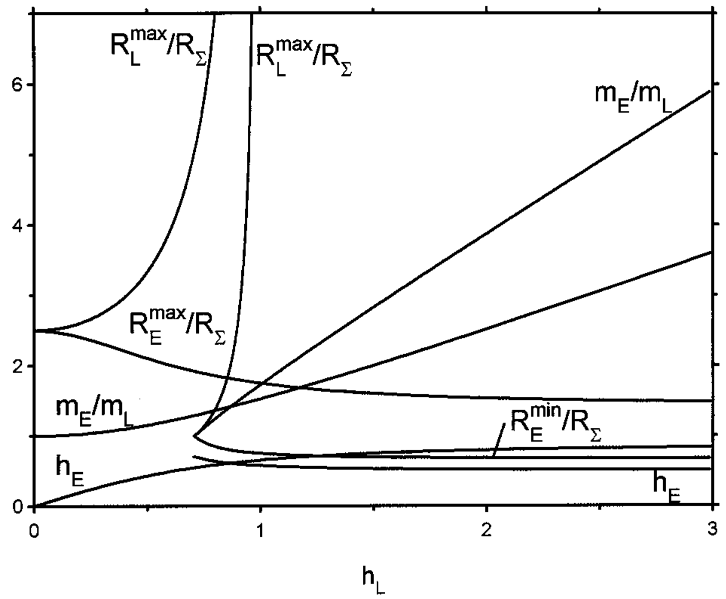

Fig. 1. Diagram illustrating properties of combined Euclidean and Lorentzian geodesic paths. The curves give values of various parameters, as functions of h_L for geodesics arriving at a constant T signature change surface at the point $R_\Sigma = 4M$(those curves starting at $h_L = 1/\sqrt{2}$), and at constant $R = 4M/5$ surface (those curves starting at $h_L = 0$). Vertical slices through the graph give the values of h_L, h_E, m_L/m_E, R_L^{max} (where it exists) and R_E^{min} or R_E^{max} for individual geodesics.

Summary

Within the Schwarzschild metric form, signature change is possible on constant R surfaces inside the horizon, but the resulting Euclidean region has strange matter, and continues to collapse to a singularity. A second signature change back to a Lorentzian region is of course possible, but only at a smaller R, closer to the future singularity. Signature change is also possible on constant T surfaces, leading into a Euclidean region which is vacuum and has geodesics which do bounce. But constant T surfaces are entirely outside the horizon, and so not very interesting, since a second transition back to a Lorenzian spacetime results in the same future as the one that was avoided.

Attempts to continue geodesics through a signature change indicated that one must match the 4-momenta, which means that particle rest masses have to jump. If a second signature change back to a Lorentzian metric occurred, the particle mass would not return to its original mass unless the model were highly symmetric.

4. The Lemaître-Tolman Case

We now shift our attention to the Lemaître-Tolman metric [40, 41], primarily because it allows us to deal simply with more general surfaces in spherical vacuum, and secondly because it makes possible a generalisation of the black hole topology to non-empty models, thus describing more realistically the collapse of matter into black holes, as well as the more standard cosmological collapse of matter, where no wormhole topology is involved [42]. This gets us closer to Smolin's scenario.

In the vacuum case, the Lemaître-Tolman metric with appropriate choice of parameters can describe the full Schwarzschild-Kruskal-Szekeres manifold, avoiding a coordinate singularity at $R = 2M$

and the accompanying change of character of the Schwarzschild R and T coordinates, and making it clear which metric element should change sign at a change of signature. Its two arbitrary functions make it much more flexible than the Kruskal-Szekeres metric.

The diagonal, synchronous, spherically symmetric metric, with an added factor of $\epsilon = \pm 1$,

$$ds^2 = \epsilon dt^2 + B^2(t,r)dr^2 + R^2(t,r)d\Omega^2 \tag{37}$$

leads to the following Einstein tensor:

$$\begin{aligned}
G_{tt} &= \frac{\epsilon(2BRR'' + BR'^2 - 2B'RR' - B^3) + (2B^2\dot{B}R\dot{R} + B^3\dot{R}^2)}{B^3R^2} && (38)\\
G_{tr} &= \frac{2(\dot{B}R' - B\dot{R}')}{BR} && (39)\\
G_{rr} &= \frac{(R'^2 - B^2) + \epsilon(2B^2R\ddot{R} + B^2\dot{R}^2)}{R^2} && (40)\\
\frac{G_{\phi\phi}}{\sin^2\theta} = G_{\theta\theta} &= \frac{R[(BR'' - B'R') + \epsilon(B^2\ddot{B}R + B^2\dot{B}\dot{R} + B^3\ddot{R})]}{B^3} && (41)\\
& && (42)
\end{aligned}$$

where $' \equiv \partial/\partial r \quad \& \quad \dot{} \equiv \partial/\partial t$ and the cosmological constant is taken to be zero.

Solving the Einstein field equations for co-moving matter, $u^\mu = \delta^\mu_t$, and zero pressure, $p = 0$, gives the Lemaître-Tolman model, and we get

$$\begin{aligned}
B^2 &= \frac{(R')^2}{1+f}, \quad f(r) \geq -1 && (43)\\
-\epsilon\dot{R}^2 &= \frac{2M}{R} + f(r) && (44)\\
\ddot{R} &= \epsilon\frac{M}{R^2} && (45)\\
8\pi\rho = G_{tt} &= -\epsilon\frac{2M'}{R^2R'} && (46)\\
R^\mu_\mu &= 2\left(\left(\frac{2B'R'}{B^3R} - \frac{(R')^2}{B^2R^2} - \frac{2R''}{B^2R}\right) + \frac{(1-\epsilon\dot{R}^2)}{R^2} - \epsilon\left(\frac{2\ddot{R}}{R} + \frac{2\dot{B}\dot{R}}{BR} + \frac{\ddot{B}}{B}\right)\right)\\
&= \frac{2M'}{R^2R'} && (47)\\
k &= R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}\\
&= 4\left(\frac{2\ddot{R}^2}{R^2} + \frac{\ddot{B}^2}{B^2} + \left(\frac{1-\epsilon\dot{R}^2}{R^2} - \frac{(R')^2}{B^2R^2}\right)^2\right.\\
&\quad \left. +2\left(\frac{R''}{B^2R} - \frac{B'R'}{B^3R} + \frac{\epsilon\dot{B}\dot{R}}{BR}\right)^2 + 4\epsilon\left(\frac{\dot{R}'}{BR} - \frac{\dot{B}R'}{B^2R}\right)^2\right)\\
&= 4\left(\frac{3M'^2}{R^4R'^2} - \frac{8M'M}{R^5R'} + \frac{12M^2}{R^6}\right) && (48)
\end{aligned}$$

where $f = f(r)$ and $M = M(r)$ are arbitrary functions of coordinate radius r, ρ is the density, and k is the Kretschmann scalar. Singularitites in k and ρ occur at $R = 0$ and $R' = 0$ regardless of ϵ. Shell crossings occur where $R' = 0$, since shells of matter at a different constant r, arrive at the same areal radius $R(t,r)$ and intersect each other. In vacuum, $M' = 0$, so ρ is zero and k is finite, and there is no physical problem, but there is a bad coordinate coverage of the space. In non-vacuum cases care needs to be taken to select the arbitrary functions which do not give rise to these physically troublesome shell crossings [43].

In the standard Lorentzian case ($\epsilon = -1$), $f(r)$ is a kind of local energy constant which determines the type of time evolution — elliptic, parabolic or hyperbolic — as well as the local geometry, and $M(r)$ represents the total effective gravitational mass within comoving radius r.

In the Euclidean case ($\epsilon = +1$) the acceleration (45) is everywhere positive, provided we select a positive 'mass' term, so a bouncing Euclidean universe can be achieved. This requires f_E to be negative in order to keep $\dot{R}$ real, i.e. $-1 \leq f_E < 0$.

We now obtain solutions to the evolution equation (44), in terms of a parameter η, and $a = a(r)$, a third arbitrary function of r, which is the time of the big bang $R = 0$, or if we use the time reverse of the following equations, the time of the big crunch. Solutions with $f \geq 0$ or $M \leq 0$ are discarded. Although we can't be sure negative 'mass' solutions are physically disallowed in a Euclidean manifold, they all reach the $R = 0$ singularity, and none of them bounce (re-expand), so they do not serve our purpose.

Lorentzian region $\epsilon = -1$: elliptic solution, $-1 \leq f_L < 0$

$$R(t,r) = \frac{M_L}{(-f_L)}(1 - \cos\eta_L), \quad t = \frac{M_L}{(-f_L)^{3/2}}(\eta_L - \sin\eta_L) + a_L(r) \tag{49}$$

Euclidean region $\epsilon = +1$: $M_E > 0$, $-1 \leq f_E < 0$

$$R(t,r) = \frac{M_E}{(-f_E)}(\cosh\eta_E + 1), \quad t = \frac{M_E}{(-f_E)^{3/2}}(\sinh\eta_E + \eta_E) + a_E(r) \tag{50}$$

Any Lemaître-Tolman model with $M' = 0$ is a vacuum model, and thus for $\epsilon = -1$ represents at least a section of the Kruskal-Szekeres-Schwarzschild space time in geodesic coordinates. However not every selection of the arbitrary functions gives complete coverage of the manifold. Novikov coordinates [44] do cover the entire manifold, and are obtained with the following choices

$$M_L = \text{const}, \quad f = \frac{-1}{1 + (r/2M_L)^2}, \quad a_L(r) = \frac{-\pi M_L}{(-f)^{3/2}} \tag{51}$$

for which the surface $t = 0$ is a simultaneous time of maximum expansion, and $f(0) = -1$ at the Schwarzschild throat, increasing monotonically to 0 as $r \to \pm\infty$. This topology — two sheets joined by a throat — may easily be extended to non-vacuum everywhere [42] by setting $M_L = M_L(r)$ with a minimum value at the throat. It is the form of $f(r)$ which determines the topology. If the asymptotic regions are closed FLRW cosmologies ($f = -kr^2$, $k = +1$), then we still expect f to rise very close to zero before decreasing again. In such dense black holes, the past and future event horizons are split, and $R = 2M_L$ is an apparent horizon [42].

Matching conditions

We perform the matching on the simplest possible surface, that of constant time, $t =$ constant. In vacuum this is merely a coordinate restriction and not a physical one, because the origin of the time coordinate, $a(r)$, is an arbitrary function of position. It amounts to finding the family of geodesics orthogonal to the transition surface, and using these as lines of constant r. The intrinsic metric of such a surface is correspondingly simple:

$$d\sigma^2 = B^2\, dr^2 + R^2\, d\Omega^2 = \frac{(R')^2}{1+f}\, dr^2 + R^2\, d\Omega^2 \tag{52}$$

When matching, a reasonable choice is to equate the angular parts, and to re-scale the coordinate radii, so that

$$\theta_E = \theta_L, \quad \phi_E = \phi_L, \quad r_E = r_L \tag{53}$$

and $[{}^3\!g_{ij}] = 0$ fixes

$$R_E = R_L = R_\Sigma, \quad B_E = B_L = B_\Sigma \tag{54}$$

Since R is continuous across the junction and is a function of r only on Σ, i.e. $R_\Sigma = R_\Sigma(r)$, we have also that

$$R'_E = R'_L = R'_\Sigma, \quad \Rightarrow f_E = f_L = f \tag{55}$$

Because of (43) the normal and the non-zero elements of the extrinsic curvature are:

$$n^\mu = \sqrt{\epsilon\epsilon_n}\,\delta^\mu_t, \qquad n^\mu n_\mu = \epsilon_n = \epsilon, \tag{56}$$

$$K_{rr} = \sqrt{\epsilon\epsilon_n}\,B\dot{B} = \frac{R'\dot{R}'}{1+f}, \qquad K_{\theta\theta} = \sqrt{\epsilon\epsilon_n}\,R\dot{R} = R\dot{R} = \frac{K_{\phi\phi}}{\sin^2\theta} \tag{57}$$

and $[K_{ij}] = 0$ leads to

$$\dot{R}_L = \dot{R}_E, \quad \dot{B}_L = \dot{B}_E \;\; \Rightarrow \;\; \dot{R}'_L = \dot{R}'_E \tag{58}$$

The surface effects are

$$E(l_\alpha) \;=\; 2\left(\frac{(R')^2}{B^2R^2} - \frac{2B'R'}{B^3R} + \frac{2R''}{B^2R} - \frac{1}{R^2}\right) = 2\left(\frac{f}{R^2} + \frac{f'}{RR'}\right) \tag{59}$$

$$B^2E(\overline{e}^r_\alpha) = E(e^\alpha_r) \;=\; 4\sqrt{\epsilon\epsilon_n}\left(\frac{8\dot{R}R'}{R^2} - \frac{9\dot{R}'}{R}\right) \tag{60}$$

$$E(n^\alpha) \;=\; \frac{2\epsilon\epsilon_n\dot{R}^2}{R^2} \tag{61}$$

The principal feature we are looking for is a bouncing universe, meaning a Lorentzian region matched to a bouncing Euclidean region that in turn may be matched to another Lorentzian region. This involves establishing the existence of at least two solution surfaces in the Euclidean region of the model under investigation. In general, given two space-like hypersurfaces, there will not be any geodesics that are orthogonal to both, so requiring both to be $t =$ constant surfaces in the same coordinate system could well be restrictive.

General transitions

Five arbitrary functions, $f(r)$, $M_E(r)$, $M_L(r)$, $a_L(r)$ and $a_E(r)$, are as yet unspecified. We now derive the necessary relations between them at a surface of signature change. We do not assume vacuum at this stage. Only models with $f < 0$ and $M_E > 0$ give rise to a Euclidean region with a bounce.

Condition (58) for $\dot{R}^2$ combines with the evolution equation (44) to give

$$M_L - M_E = \dot{R}^2_\Sigma R_\Sigma \qquad \text{or} \qquad M_L + M_E = -fR_\Sigma \tag{62}$$

The sign of $\dot{R}$ must still be matched.

Inserting the parametric expressions for R_Σ (49) and (50) into (62) gives

$$\cos\eta_{L\Sigma} = -\frac{M_E}{M_L}, \qquad \cosh\eta_{E\Sigma} = \frac{M_L}{M_E} \tag{63}$$

$$\Rightarrow \quad M_E \le M_L \;, \qquad -1 \le \;\; \cos\eta_L \;\; \le 0 \;, \qquad 1 \le \cosh\eta_E \le \infty \tag{64}$$

which, combined with the continuity of $\dot{R}$ for collapsing models, yields

$$t_\Sigma < 0: \quad \eta_{L\Sigma} = 2\pi - \cos^{-1}(-M_E/M_L), \quad \eta_{E\Sigma} = -\cosh^{-1}(M_L/M_E) \tag{65}$$

Since the Euclidean region doesn't have arbitrarily large $\dot{R}$, the transition cannot happen arbitrarily close to $R = 0$. On the transition surface t_Σ is constant, so $\eta_{L\Sigma}$ and $\eta_{E\Sigma}$ are functions of r only. Thus

$$t_{E\Sigma} = t_\Sigma \;=\; a_E(r) + \frac{M_E}{(-f)^{3/2}}(\sinh\eta_{E\Sigma} + \eta_{E\Sigma}) \tag{66}$$

$$t_{L\Sigma} = t_\Sigma \;=\; a_L(r) + \frac{M_L}{(-f)^{3/2}}(\eta_{L\Sigma} - \sin\eta_{L\Sigma}) \tag{67}$$

If possible, we choose $a_E(r) = 0$, so that (66) gives a time-symmetric coordinate coverage in the Euclidean region. This permits a second copy of any transition surface found away from $t = 0$, and thus ensures a bounce. To obtain a specific solution, we fix t_Σ and any two of M_L, M_E, a_L, a_E, to obtain the others.

Further, (62) plus the requirement that the transition surface be inside the external horizon $R = 2M_{L\,max}$, where $M_{L\,max}$ is the total exterior mass of the collapsing cloud, gives us

$$f(r) < -\frac{M_L(r) + M_E(r)}{2M_{L\,max}} < -\frac{M_L(r)}{2M_{L\,max}} \tag{68}$$

since $0 \le M_E \le M_L$. At the centre we have $f(0) < -(M_{L\,min}/2M_{L\,max})$ and outside the cloud or at large r, $f(r > r_{max}) < -(1/2)$, where r_{max} is the smallest radius for which $M_L(r) = M_{L\,max}$. This is a very stong restriction on f.

Vacuum to Vacuum

We set the mass term constant (hence $\rho = 0$) and equal on either side $M_L = M_E$. By (62) this gives $\dot{R}_\Sigma = 0$, and by (49) and (50) the areal radius can only be matched at

$$\eta_{E\Sigma} = 0, \quad \eta_{L\Sigma} = \pi, \quad R_\Sigma = \frac{2M}{(-f)}, \quad -1 \le f \le 0 \tag{69}$$

the loci of minimum and maximum expansion of the Euclidean and Lorentzian coordinates respectively. Clearly, this case is equivalent to a constant T transition in the Schwarzschild case, as Σ touches $R = 2M$ but otherwise lies entirely outside the horizon. If we set $a_E = 0$, to obtain a symmetric coverage of the Euclidean region, we have that the transition time is $t_\Sigma = 0$ — i.e. there is no 'time' between the two transitions. This confirms our earlier suspicion that minimum expansion at the middle of the throat is also a unique event in the Euclidean Schwarzschild topology.

Retaining M_E and M_L constant, but not necessarily equal, (63) shows that both η_E and η_L are constant on the transition surface. The parametric expressions for t (66)-(67) then establish the relation between $f(r)$ and $a(r)$. Again symmetric coverage of the Euclidean region — $a_E = 0$ in (66) — would require $f =$ constant and hence $R_\Sigma =$ constant. The only way to get surfaces which have constant t, R, and f, is to set $f = -1$ — dealt with next.

Constant R

The constant t surfaces can also be made constant R surfaces, thus yielding the closed Kantowski-Sachs model [45] in Lemaître-Tolman coordinates [46]. This is done by setting

$$M = M_1 \int \sqrt{1+f}\; dr + M_0, \quad a = a_1 \int \sqrt{1+f}\; dr + a_0, \quad M_0, M_1, a_0, a_1 \text{ constants} \tag{70}$$

and then taking the limit $f \to -1$, leading to

$$8\pi\rho = -\epsilon \frac{2M_1}{R^2 B} \tag{71}$$

$$\epsilon = -1: \quad R_L = M_{0L}(1 - \cos\eta_L), \quad B_L = 2M_{1L} - (M_{1L}\eta_L + a_{1L})\frac{\sin\eta_L}{(1-\cos\eta_L)} \tag{72}$$

$$\epsilon = +1: \quad R_E = M_{0E}(\cosh\eta_E + 1), \quad B_E = 2M_{1E} - (M_{1E}\eta_E + a_{1E})\frac{\sinh\eta_E}{(\cosh\eta_E + 1)} \tag{73}$$

In Lorentzian vacuum, $M_1 = 0, \epsilon = -1$, these coordinates only cover $R \le 2M_L$, and may be similarly incomplete in dense models. Shell crossings may be avoided for $-2\pi < (a_{1L}/M_{1L}) < 0$ in Lorentzian regions, but not in Euclidean regions. However, for $a_{1E} = 0$, shell crossings are removed if transitions happen at $\mid \eta_{E\Sigma} \mid < 2.3994$ which is the positive root of $2(\cosh\eta_E + 1) - \eta_E \sinh\eta_E = 0$.

By (62), $R_\Sigma = M_{0E} + M_{0L}$ and for $M_{0E} > 0$ and Σ hidden ($R_\Sigma \le 2M_{0L}$) we need $M_{0L} < R_\Sigma \le 2M_{0L} \Rightarrow 0 < M_{0E} \le M_{0L}$. Equations (62)-(63) and (66)-(67) become

$$R_\Sigma = M_{0E} + M_{0L}, \quad \cos\eta_{L\Sigma} = -\tfrac{M_{0E}}{M_{0L}}, \quad \cosh\eta_{E\Sigma} = \tfrac{M_{0L}}{M_{0E}} \tag{74}$$

$$a_{0E} + M_{0E}(\sinh\eta_{E\Sigma} + \eta_{E\Sigma}) = t_\Sigma = a_{0L} + M_{0L}(\eta_{L\Sigma} - \sin\eta_{L\Sigma}) \tag{75}$$

and because the matching of R_Σ and $\dot{R}_\Sigma$ no longer ensures B_Σ and $\dot{B}_\Sigma$ is matched, we get two extra conditions

$$\begin{gathered}(M_{0E}M_{1L} - M_{0L}M_{1E})R_\Sigma + \\ ((M_{1L} - M_{1E})t_\Sigma - a_{0L}M_{1L} + a_{0E}M_{1E} + a_{1L}M_{0L} - a_{1E}M_{0E})\sqrt{M_{0L}^2 - M_{0E}^2} = 0\end{gathered} \tag{76}$$

$$(M_{1L} + M_{1E})t_\Sigma - a_{0L}M_{1L} - a_{0E}M_{1E} + a_{1L}M_{0L} + a_{1E}M_{0E}) = 0 \tag{77}$$

Simplifications are obtained by requiring a symmetric Euclidean region $a_{0E} = 0$, $a_{1E} = 0$. Vacuum to vacuum is not possible since $M_{1L} = 0$, $M_{1E} = 0$ implies $a_{1E} = 0 = a_{1L}$, and thus $B_L = 0 = B_E$ at all times. Similarly, vacuum to non-vacuum is not possible. The dense models are highly symmetric, as they have uniform density on constant R surfaces. A sample set of values are: $M_{0E}/M_{0L} = 0.1$, $\eta_{E\Sigma} = -2.9932$, $\eta_{L\Sigma} = 4.6122$, $a_{0E} = 0$, $t_\Sigma/M_{0L} = -1.2943$, $a_{0L}/M_{0L} = -6.9015$, $R_\Sigma/M_{0L} = 1.1$, $a_{1E} = 0$, $M_{1E}/M_{1L} = 0.0407$, $a_{1L}/M_{1L} = -5.5476$, $B_\Sigma/M_{1L} = 1.1539$, $\rho_L M_{0L}^2 = 0.0570$, $\rho_E M_{0L}^2 = 0.0057$.

Dust to Dust — Wormhole Topology

We set $M_L = M_L(r)$ and $M_E = M_E(r)$. Since space-time is no longer empty, not all geodesic coordinate systems are equivalent to the comoving one, so a symmetric coordinate coverage of the Euclidean region becomes essential to ensure the existence of a second transition surface. From (63) and (66) with $a_E = 0$ we obtain:

$$F(r) \equiv \frac{(-f)^{3/2} t_\Sigma}{M_L} = \frac{\sinh\eta_{E\Sigma} + \eta_{E\Sigma}}{\cosh\eta_{E\Sigma}} \equiv D(\eta_{E\Sigma}) \tag{78}$$

where the right hand side defines $D(\eta_{E\Sigma})$, and the left hand side defines $F(r)$. For a Kruskal-Szekeres-Schwarzschild type topology (Lorentzian) [42], we expect f to take a Novikov-like form, i.e. symmetric, $f(-r) = f(r)$, with $f(0) = -1$ a minimum at $r = 0$, and rising monotonically. To cover the asymptotically flat regions at large R requires $f(\pm\infty) = 0$ (since $f_E > 0$ doesn't give a bounce). We expect the mass M_L to be minimum at $r = 0$ and rising monotonically to a finite value. For example

$$f = \frac{-1}{1 + r^2}, \quad M_L = \frac{M_{L\,min} + M_{L\,max} r^2}{1 + r^2} \tag{79}$$

With these choices and $M_{L\,min} = (1/3)M_{L\,max}$, $F(r)$ and $D(\eta_{E\Sigma})$ are plotted in Figs 2a and 2b. The main features of $F(r)$ are dictated by the topology, and are independent of the particular choices of $f(r)$ and $M_L(r)$.

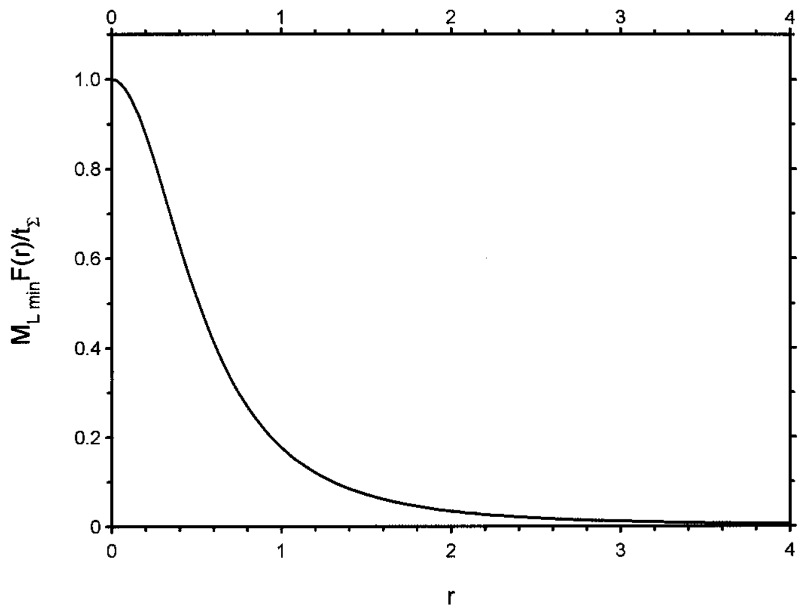

Fig. 2a. The function $F(r)$ vs. r.

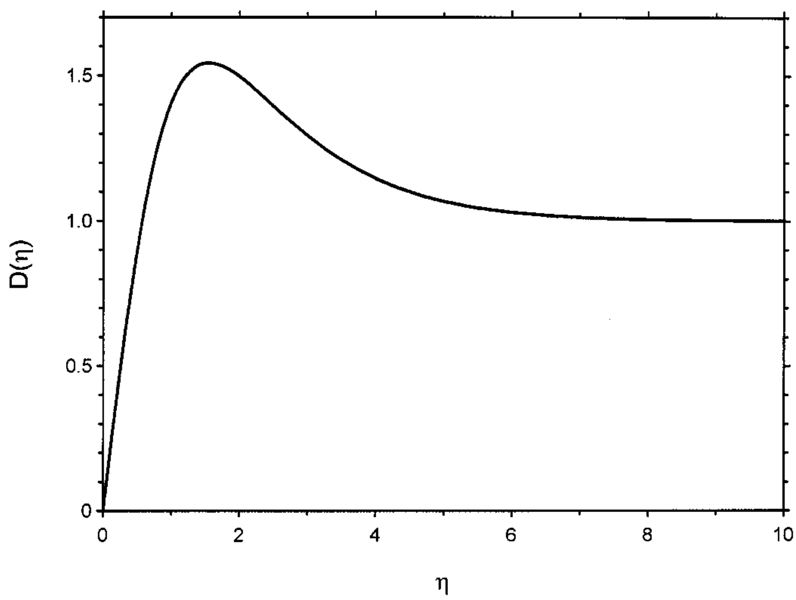

Fig. 2b. The function $D(\eta)$ vs. η.

The mapping between F and D is needed to fix the r dependence of $\eta_{E\Sigma}(r)$. We have at our disposal only one constant, $t_\Sigma/M_{L\,min}$, which is freely adjustable, so we need to select the section of the D graph which includes 0, in order to accomodate $f \to 0$. Also since F is monotonic, the range of $\eta_{E\Sigma}$ cannot extend through the maximum of $D(\eta_{E\Sigma})$, D_{max}. Hence we have a restriction on when in the Euclidean evolution the transition can occur. To obtain $f(0) = -1$ we need

$$f = -1 \quad \Rightarrow \quad \frac{t_\Sigma}{M_{L\,min}} = \frac{\sinh \eta_{E\Sigma} + \eta_{E\Sigma}}{\cosh \eta_{E\Sigma}} \tag{80}$$

and consequently the following ranges are allowed:

$$0 \leq \frac{t_\Sigma}{M_{L\,min}} \leq D_{max}, \quad 0 \leq \eta_{E\Sigma} \leq \eta_{E\Sigma max}, \quad \frac{M_L}{\cosh \eta_{E\Sigma max}} \leq M_E \leq M_L \tag{81}$$

where

$$D_{max} = 1.5434, \qquad \text{at} \qquad \eta_{E\Sigma max} = 1.5434 = D_{max} \tag{82}$$

Although the range $\eta_{E\Sigma max} \leq \eta_{E\Sigma} \leq \infty$ is not obviously precluded in principle, it results in a different sign for $d\eta_{E\Sigma}/dr$, which affects R'_Σ, $\rho_{E\Sigma}$ and $\rho_{L\Sigma}$, as well as limiting the range of R.

We now find some sample solutions numerically. Our plotting procedure is as follows:

— Select the Lorentzian mass, M_L;
— Choose a transition time t_Σ which complies with (81);
— Generate values of M_E which span all values allowed by (81), for the given t_Σ and M_L;
— For each M_E value calculate:
  — f from (66): $f = -[(\sinh \eta_{E\Sigma} + \eta_{E\Sigma})M_E/(t_\Sigma - a_E)]^{2/3}$;
  — r from (79): $r = \sqrt{(1+f)/(-f)}$;
  — R_Σ from (62): $R_\Sigma = (M_L + M_E)/(-f)$;
  — a_L from (67): $a_L = t_\Sigma - (\eta_{L\Sigma} - \sin \eta_{L\Sigma})M_L/(-f)^{3/2}$;
  — $\rho_{E\Sigma}$ and $\rho_{L\Sigma}$ from (46): $\rho_\Sigma = 2M'/R_\Sigma^2 R'_\Sigma$

(The last requires the values of M'_E and R', obtained from the derivatives with respect to r of (66), (67), (63), and (62)).

Having found two surfaces where a signature change could occur, i.e. $t_\Sigma^{L\to E} = -t_\Sigma^{E\to L} > 0$, the idea is to excise the future singularity in one Lorentzian region, and the past singularity in the other, and join them with the Euclidean region.

The following three models are typical. They use the forms $M_L = (M_{L\,min} + M_{L\,max} r^2)/(1+r^2)$, $(-1 + f_\infty r^2)/(1+r^2)$ and the values:

(a) $t_\Sigma/M_{L\,min} = 1.4096484 = D(1)$
  $M_{L\,min}/M_{L\,max} = 0.93$
  $f_\infty = -0.93$
(b) $t_\Sigma/M_{L\,min} = 1.5434 = D_{max}$
  $M_{L\,min}/M_{L\,max} = 0.93$
  $f_\infty = -0.93$
(c) $t_\Sigma/M_{L\,min} = 1.4096484 = D(1)$
  $M_{L\,min}/M_{L\,max} = 1/3$
  $f_\infty = 0$

Figs. 3 to 6 show f, a_L, M_E, M_L, R_Σ, $\rho_{E\Sigma}$ and $\rho_{L\Sigma}$ as functions of r, for these models.

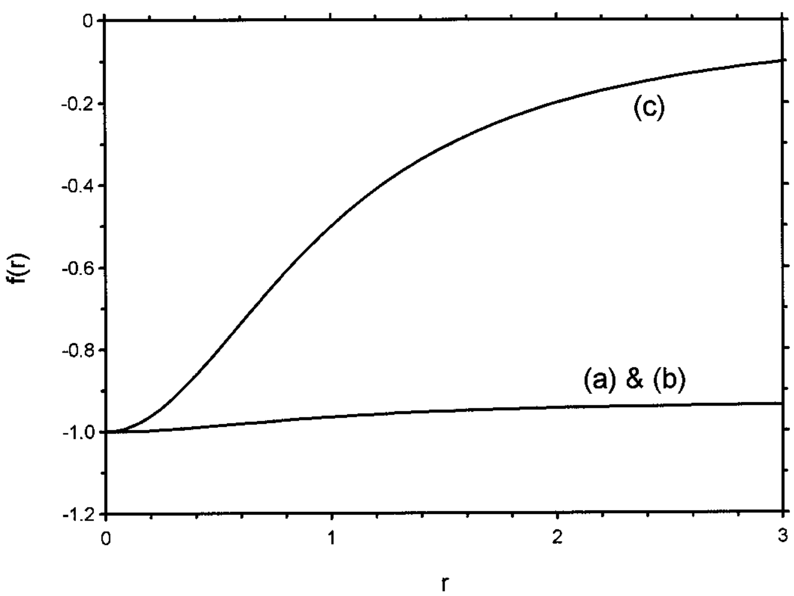

Fig. 3. $f(r)$ vs. r for the three Lemaître-Tolman signature change models, (a), (b) and (c).

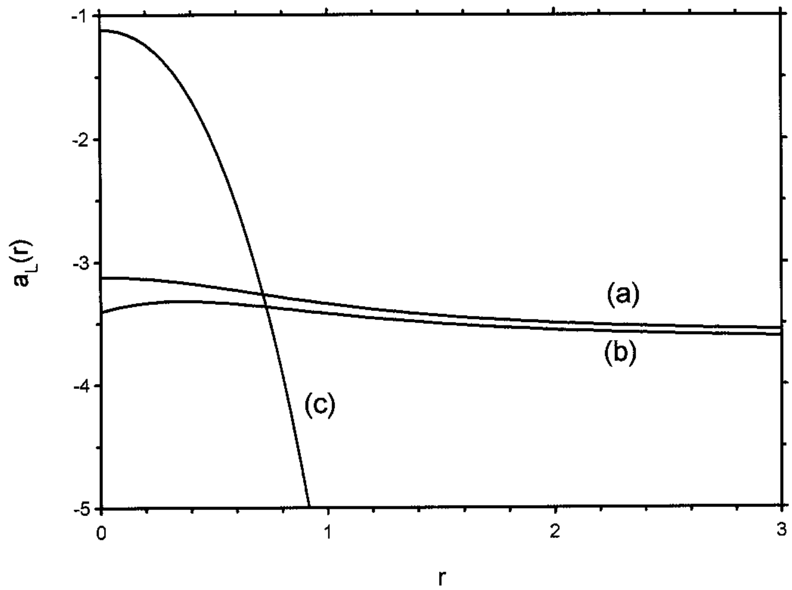

Fig. 4. $a_L(r)$ vs r, for models (a), (b) and (c). Note that (b) has its maximum away from $r = 0$, whereas (a) and (c) has it at $r = 0$.

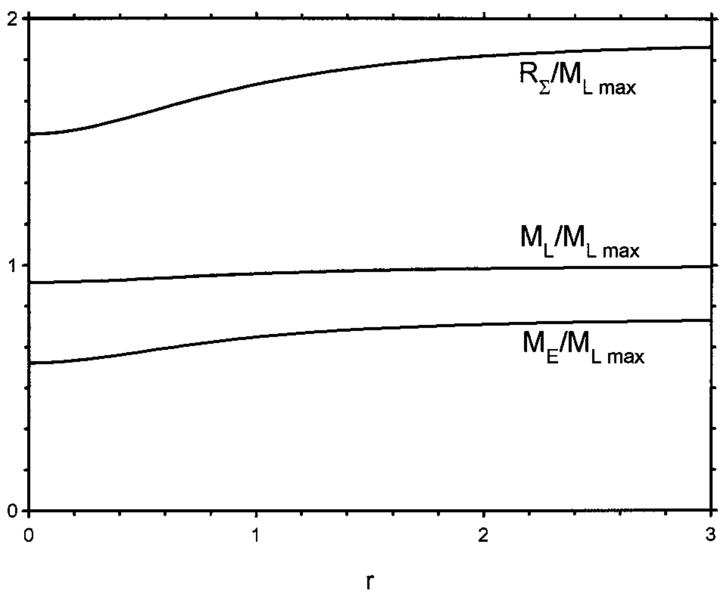

Fig. 5a. The run of $R_\Sigma(r)$, $M_L(r)$ and $M_E(r)$ for model (a).

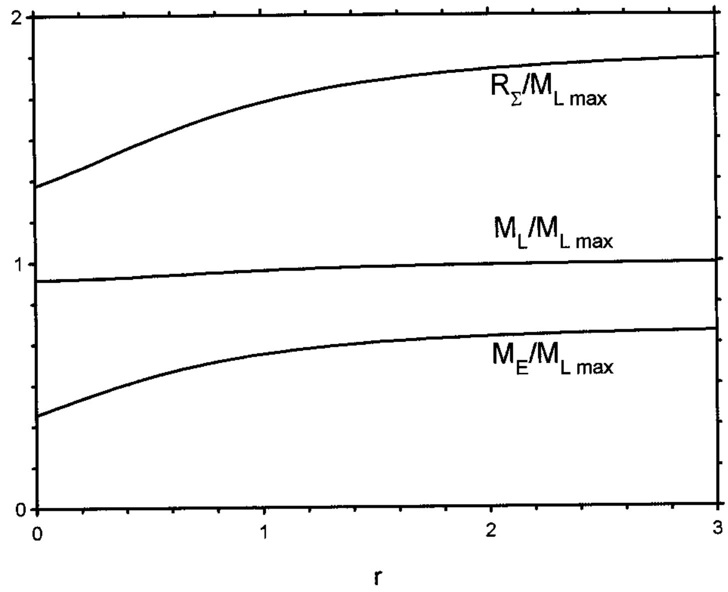

Fig. 5b. $R_\Sigma(r)$, $M_L(r)$ and $M_E(r)$ for model (b). Note that none of these quantities are smooth through the origin $r = 0$.

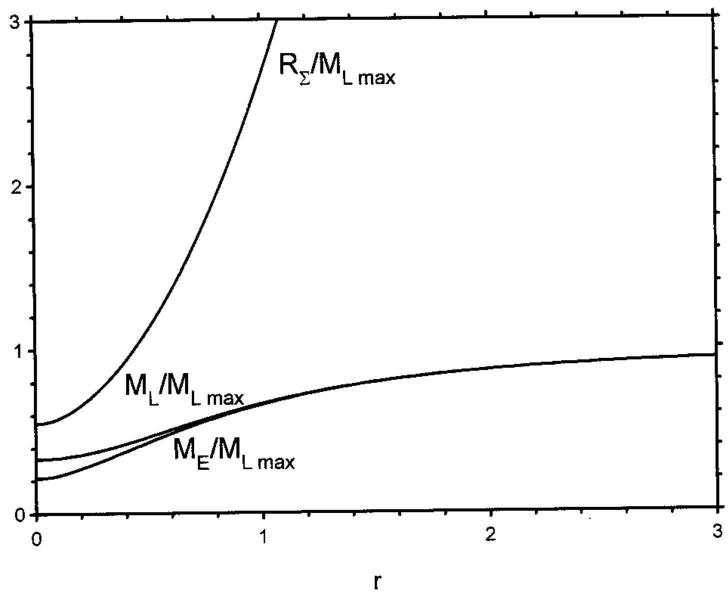

Fig. 5c. $R_\Sigma(r)$, $M_L(r)$ and $M_E(r)$ for model (c).

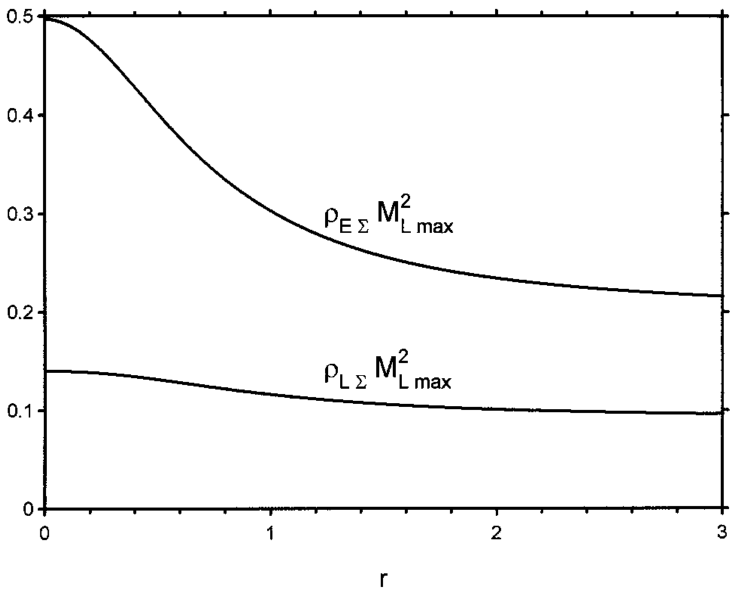

Fig. 6a. The run of ρ_L and ρ_E for model (a).

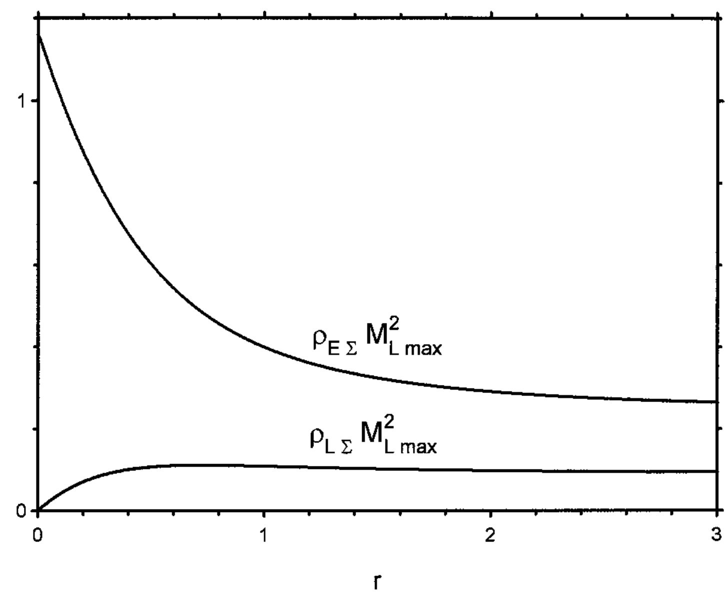

Fig. 6b. ρ_L and ρ_E for model (b). Note that neither quantity is smooth through the origin $r = 0$.

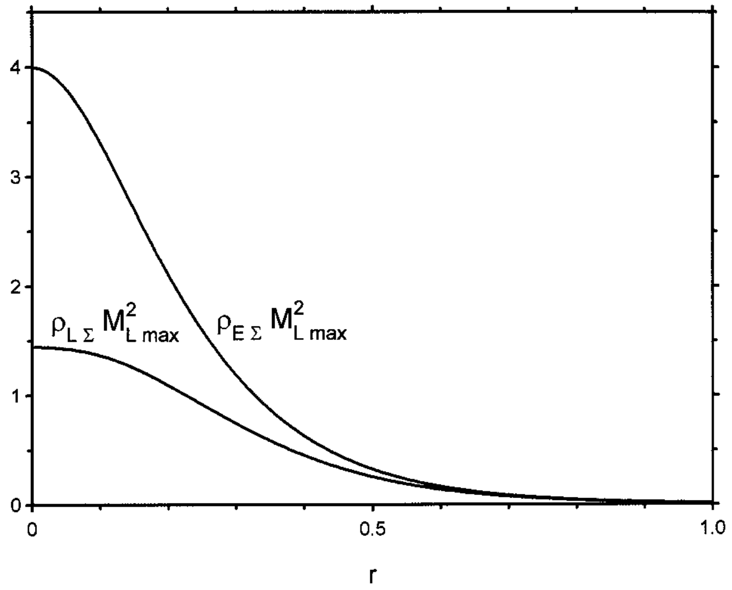

Fig. 6c. ρ_L and ρ_E for model (c).

As expected, the areal radius has a minimum at $r = 0$ and does not go singular, and both the Lorentzian and Euclidean 'densities' are well behaved at the transition.

Model (a) has Σ entirely inside the external event horizon $R = 2M_{L\,max}$, as shown in Fig. 5a, and it was found that there must be very little variation in $M_L(r)$ and $f(r)$ (Fig. 3) to achieve this. The densities on each side of Σ are also only mildly varying, as shown in Fig. 6a. It could be thought of as a perturbation of a Kantowski-Sachs model, allowing the particle world lines to emerge beyond $R = 2M_L$ briefly, before recollapsing back inside and encountering the signature change. Since this is true even for $r \to \infty$, the particles do not fill the spacetime, and the model may be completed by matching to a vacuum exterior. This makes a very satisfactory model of signature change in the black hole topology.

Model (b) differs only in having the largest possible value of $t_\Sigma / M_{L\,min}$. This results in a_L, M_E, M_L, R_Σ, $\rho_{E\Sigma}$ and $\rho_{L\Sigma}$ all having non-zero gradient at $r = 0$, meaning these quantities are discontinuous through the origin — see Figs. 5b and 6b. A feature of this model is that the bang time $a_L(r)$ has a maximum away from $r = 0$ (Fig. 4), indicating the particle world lines self intersect somewhere in the time evolution of the Lorentzian part of the model [43, 42]. This 'shell crossing' is now a serious deficiency of the model, involving densities that diverge and go negative. Otherwise it is very similar to (a).

Model (c) is complete since $r \to \infty$ covers the asymptotic regions of the model, and $f \to 0$ means it is asymptotically flat. However the signature change surface passes out of $R = 2M_{L\,max}$ and extends to $R = \infty$ (Fig. 5c), which is not ideal, despite the nice density profile in Fig. 6c.

Summary

Within the Lemaître-Tolman metric form, the wormhole topology is possible in both Lorentzian and Euclidean regions with and without matter (dust) present. The Euclidean region bounces provided the mass function is positive (as defined in (44)), so negative mass models were not considered. Only constant t transition surfaces, which are orthogonal to the fluid flow, were considered.

Vacuum to Vacuum signature transitions are equivalent to the uninteresting constant T Schwarzschild case.

Signature transitions *are* possible inside the horizon if the density is non zero on both sides of the transition. Time symmetric Euclidean regions permit a second transition following the bounce, emerging into an expanding spacetime behind the past singularity. This is true both for the constant R transitions in uniform density Kantowski-Sachs type models, and the more general inhomogeneous case. The general case is particularly satisfactory as it models a finite cloud of dust.

5. Conclusion

We have succeeded in demonstrating the possibility that a change in the signature of space-time may occur in the late stages of black hole collapse, resulting in a Euclidean region which bounces and re-expands, passing through a second signature change to a new expanding Lorentzian space-time. The classical singularity at $R = 0$ is thus avoided. Such transition surfaces necessarily have non-zero extrinsic curvature.

The model of signature change employed here is strictly classical. Quantum cosmological questions, for example the relative probability of different sorts of transitions, have not been considered. We have based our notion of manifold continuity on the fulfillment of the Darmois type matching conditions, since they are invariant to the coordinates used, and no modifications are necessary to adapt them to surfaces of signature change. As discussed in [19], surface effects appear in the conservation laws, even when stronger conditions than Darmois' are imposed.

Based on this approach, we have shown that signature transitions are possible in a spherically symmetric Lorentzian space-time, in both the Schwarzschild and Lemaître-Tolman metric representations, though the ensuing Euclidean region might not be empty. Once the Israel identities are adapted to signature change, continuous 'density' is no longer required.

Within the Schwarzschild metric form, such a transition was possible on a constant T slice, but this can only span the outer region $R \geq 2M$. Conversely the constant R surface can be entirely inside the horizon, but does not lead to a bouncing Euclidean region. Thus these models are not satisfactory.

A study of the geodesics in each region showed that the two Euclidean regions, $R \geq 2M$ and $R \leq 2M$ were in fact complete manifolds. It was found necessary to match geodesic 4-momenta, P^μ, at the signature change, in order that all geodesics could be continued. This naturally means P_μ and $m^2 = |P^\mu P_\mu|$ are discontinuous. This is consistent with the fact that the density can jump at a signature change.

These results were generalised using constant t transitions in the Lemaître-Tolman metric form. With suitable choices of the function $f(r)$, this metric can reproduce the Kruskal-Szekeres topology of two sheets joined by a wormhole, but with non zero density. It also has a Kantowski-Sachs limit. It was found possible to have a signature change surface completely hidden inside the horizon $R = 2M$ in the Lorenzian region, provided there was non-zero density in both the Lorentzian and Euclidean regions. In the Lorentzian region, the matter is of finite extent, and may be surrounded by vacuum. It was also possible for the Euclidean region to be time-symmetric, so that after the bounce, the matter expands through a second signature change into another Lorentzian region — a new universe. This makes a very satisfactory model of collapse, bounce and re-expansion of a mass concentration. Within the Lemaître-Tolman form, a constant t signature change surface cannot be arbitrarily close to the Lorentzian singularity $R = 0$. One might expect such transitions to occur only a Planck time before $R = 0$, which would require us to consider a different equation of state in the Euclidean region. This may well relax the limits on η_Σ and f that were found.

It was found possible to hide the entire signature change surface inside the Lorentzian horizon $R = 2M_L$, if the model is non-vacuum in the central regions, with a vacuum exterior. The matter is collapsing from not far outside the horizon, as may be expected for a collapsing compact object. The limit on f for a completely hidden surface implies (1) that all the infalling matter must be in a finite cloud, moving on tightly bound paths ($R_{max} \leq 4M_L$), surrounded by vacuum, and (2) that the black hole topology is required. This provides a classical bounce model of the kind we sought. A quantum cosmological analogue could be of interest in the context of Smolin's 'natural history' of universes proposal. A treatment similar to that of Kerner and Martin [47, 48] could permit the non-zero extrinsic curvature that is required.

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