

# Finding The Metric of the Cosmos

Charles Hellaby

Mathematics of Gravitation II, Banach Centre, Warsaw, 3/9/2003

## Abstract

Current measurements are finally providing reasonably reliable estimates of the cosmological parameters of a Robertson-Walker Model. Up to now a homogeneous model of the Cosmos has been assumed, but this only describes the universe on the very largest scales. On smaller scales the structure is very inhomogeneous and non-linear.

With the new generation of Astronomical surveys, current and planned, we shall be mapping the matter in the Universe in great detail. This opens up the possibility of determining the detailed geometry of our Cosmos (i.e. its metric) from the Einstein equations.

I review work on observational relations in inhomogeneous cosmological models, and on how to extract metric information from observations.

## 1 Two Cosmic Surveys

Two recent automated redshift surveys greatly extend our map of the matter distribution in the universe, and hint at the vast amount of data we can expect in the near future. This data will make possible a detailed determination of the geometry of spacetime — the metric of the cosmos.

### SDSS — Sloan Digital Sky Survey — 2000-2005

The logo shows the sky coverage

— 1/4 of the sky

— a large cap in the galactic north + 3 slices in the south.

#### Aim:

Systematically map one-quarter of the entire sky,

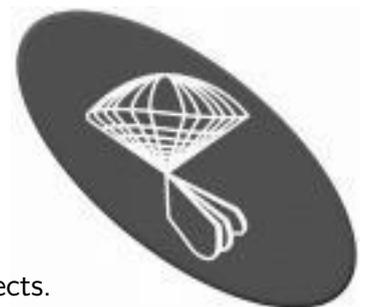
Produce a detailed image

Determine positions and absolute brightnesses of more than  $10^8$  celestial objects.

Measure the distance to  $10^6$  of the nearest galaxies, (vol 100 times previous)

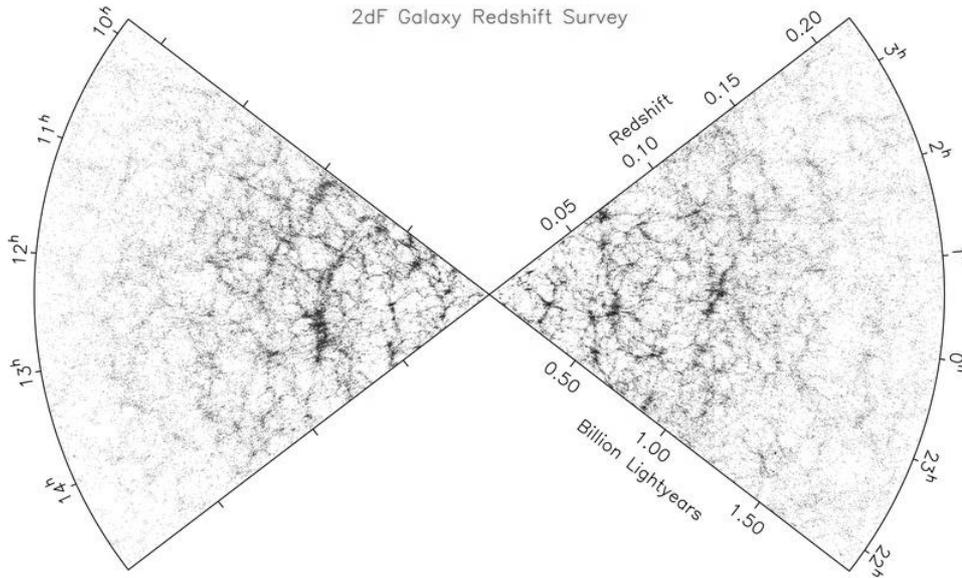
Record distances to  $10^5$  quasars.

Public data release 1 — 2003 — <http://www.sdss.org/dr1/>



## 2dF — 2 Degree Field — Galaxy Redshift Survey

A deep survey of a pair of thin ( $2^\circ$ ) wedges in opposing sky directions.



**Aim:**

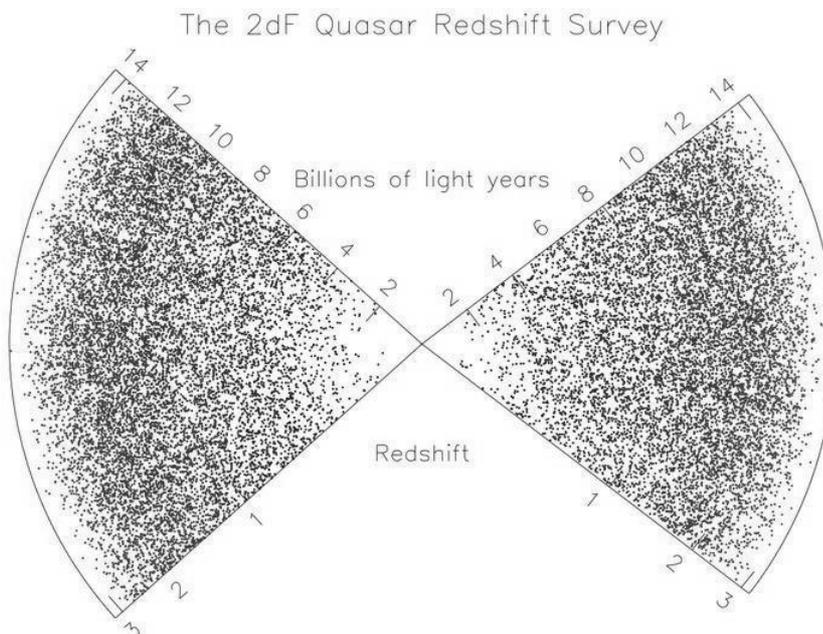
Redshift survey of 250,000 galaxies

Make a 3D map of the Southern Sky. Joint UK-Australian project.

Finished (April 2002), with over 220,000 galaxy redshifts obtained.

Public Data release — 2 DVDs — July 2003.

## 2dF — 2 Degree Field — Quasar Redshift Survey



**Aim:**

Take spectra of 30,000 QSOs in parallel.

Integrated with the galaxy survey.

Completed with over 23,000 redshifts obtained.

## 2 Theory & Observation

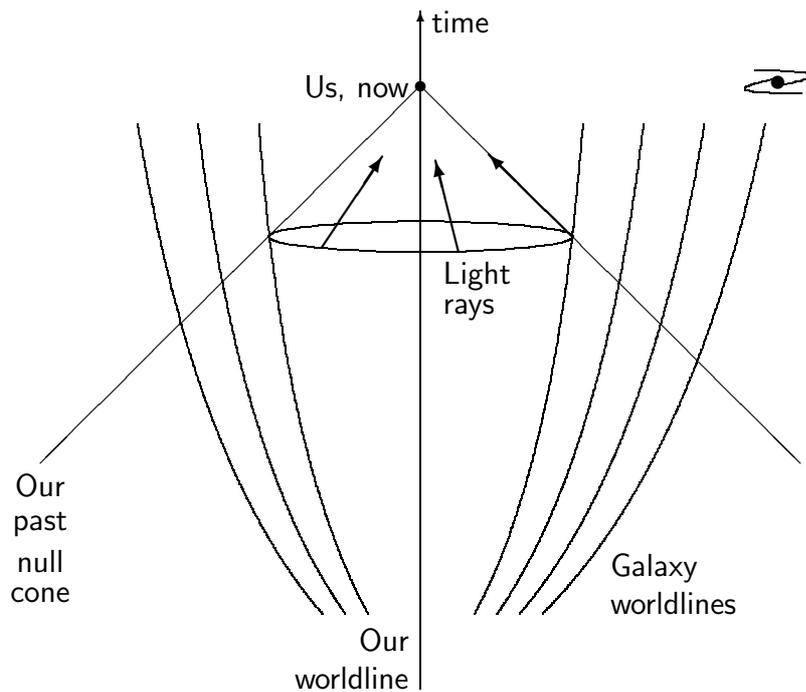
The conventional wisdom in cosmological modelling is that the Copernican principle ensures homogeneity on the very large scale. However, this scale is still not well defined, and indeed it is more realistic to talk about the degree of inhomogeneity as a decreasing function of the averaging scale. With enough data on the matter distribution, we should be able to quantify homogeneity — and thus check the Copernican assumption — much better.

Theory	Supporting Observations
Assume: EFEs .....  Homogeneity ..... Isotropy .....	Solar system, binary pulsar, CMB, nucleosynthesis  Not observed  Galaxy surveys, COBE
Matter content & Eq of state — adjustable	Favoured theory says not all matter observed/observable.
<div style="text-align: center;">↓</div> FLRW model Parameters $H_0, k, \Lambda, p(\rho)$ [ or $\Omega_k, \Omega_\Lambda, \Omega_b, \Omega_{CDM}, \dots$ ]	Primary observables: For each galaxy: <ul style="list-style-type: none"> <li>• redshift <math>z</math></li> <li>• apparent luminosity <math>\ell(z)</math></li> <li>• angular diameter <math>\delta(z)</math></li> <li>• number density in redshift space <math>n(z)</math></li> </ul>
Model predicts: <ul style="list-style-type: none"> <li>• Luminosity distance <math>d_L(z)</math></li> <li>• Diameter distance <math>d_D(z)</math></li> <li>• Mass density <math>\rho(z)</math></li> </ul> Should agree with observations: $d_L(z) = \sqrt{L/\ell}$ $d_D(z) = \sqrt{D/\delta}$ etc.	Not directly observable: <ul style="list-style-type: none"> <li>• absolute luminosity <math>L</math></li> <li>• Actual diameter <math>D</math></li> <li>• Mass of sources <math>m</math></li> </ul> These need other observations and theoretical input.

It is notable theoretical quantities ( $d_L, d_D, \text{etc}$ ) do not relate directly to observables ( $\ell, \delta, \text{etc}$ ). Each relation involves a source evolution function ( $L, D, \text{etc}$ ) — an intrinsic property of the galaxies that evolves with time, and which must be known independently.

Some studies of source evolution have deduced the evolution function by assuming the difference between observations and RW predictions are due to evolution. This removes the possibility of verifying homogeneity.

### 3 Observer's Null Cone



#### Advantage/Problem

greater distance = longer ago

If far away objects look different ..... is it because of

- Cosmic evolution (EFEs): the equation of state determines the scale factor evolution which determines redshift down the null cone,

$$p(\rho) \rightarrow S(t) \rightarrow z$$

- Source evolution: the same type of object looks different long ago (at large  $z$ ) then recently (small  $z$ ),

$$L = L(t), \quad D = D(t), \quad m = m(t)$$

- or Inhomogeneity: spatial variation makes source properties different far away (large  $z$ ) and nearby (small  $z$ ),

$$L = L(r), \quad D = D(r), \quad m = m(r) \rightarrow \rho = \rho(t, r)$$

For example:

If sources are brighter at large  $z$ ,

this COULD be due to a homogeneous universe with source evolution, ...

OR it COULD be due to a homogeneous universe with exotic matter content, ...

OR it COULD be due to an inhomogeneous universe with spatial variation in source properties.

For example:

If sources are bluer at large  $z$

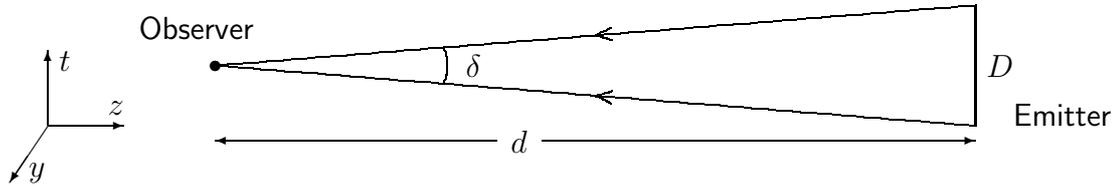
this COULD be due to a homogeneous universe with source colour evolution, ...

OR it COULD be due to an inhomogeneous universe creating spatial variation in source colours.

## 4 Diameter Distance

If we know the true diameter  $D$  of a source, we can get its distance from its angular diameter  $\delta$ :

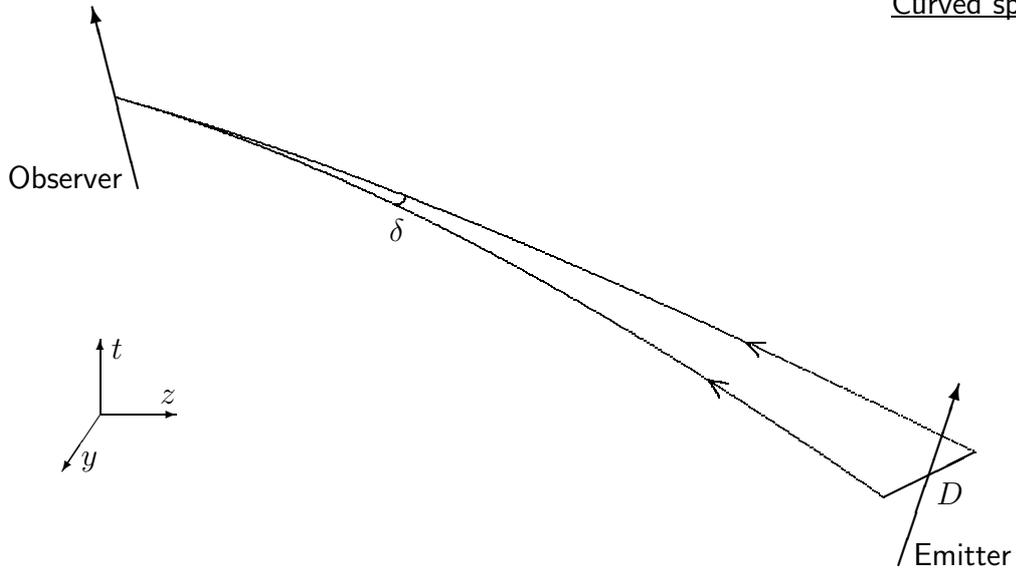
Flat space



$$d = \frac{D}{\delta} \quad \text{for } d \gg D$$

$$= \sqrt{\frac{A}{\omega}}, \quad A = \text{true area}, \quad \omega = \text{solid angle}$$

Curved space-time



Proper distance is not a useful measure — it is zero down the null cone, and spatial proper distances change with time.

In curved space, we define  $d_D$  by the same equation

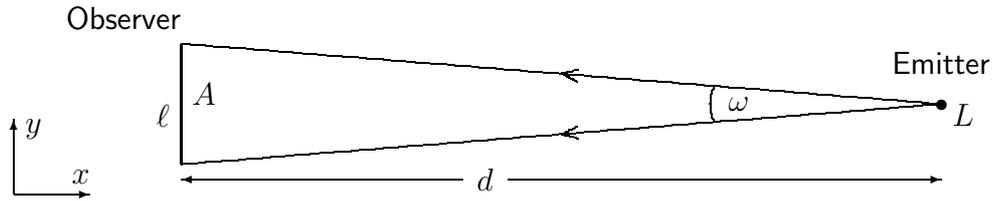
$$d_D = \frac{D}{\delta}$$

and then, for any given model, calculate the expected  $d_D(z)$ .

# 5 Luminosity Distance

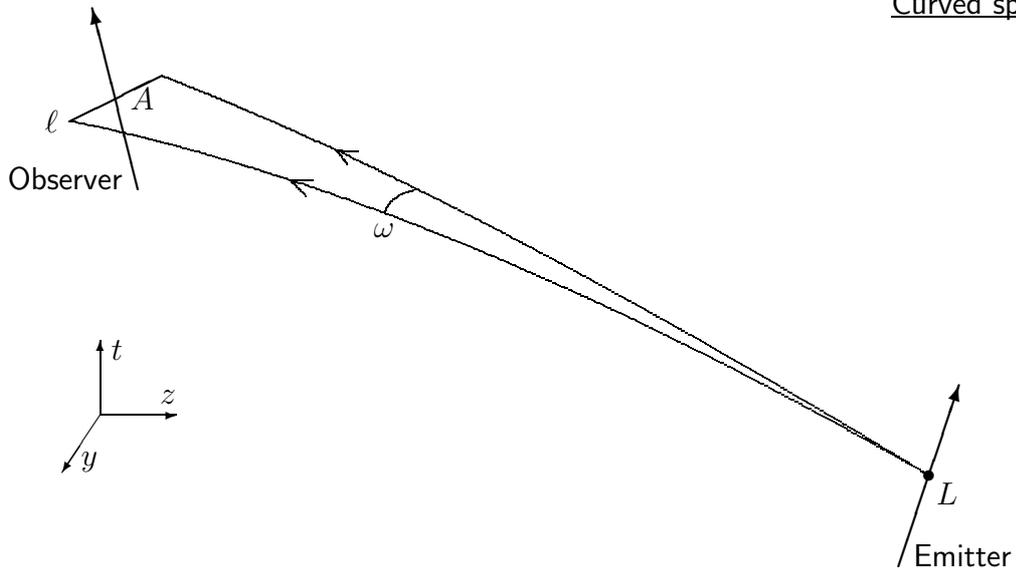
If we know the absolute luminosity  $L$  of a galaxy, we can find its distance from its apparent (measured) luminosity  $\ell$ :

Flat space



$$d = \sqrt{\frac{L}{\ell}} \quad \text{for } \omega \ll 4\pi$$

Curved space-time



In curved space, we define  $d_L$  by the same equation

$$d_L = \sqrt{\frac{L}{\ell}}$$

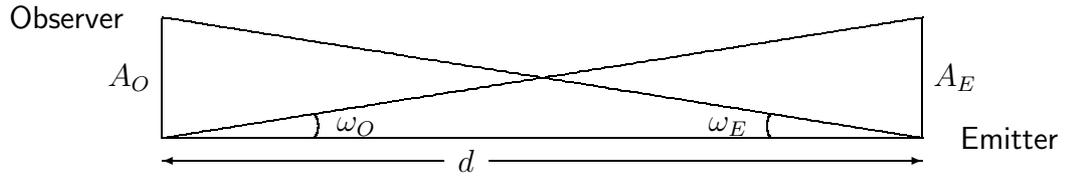
and then, for any given model, calculate the expected  $d_D(z)$ .

How do these two distances relate?

$$d_D \overset{?}{\leftrightarrow} d_L$$

# 6 Reciprocity Theorem

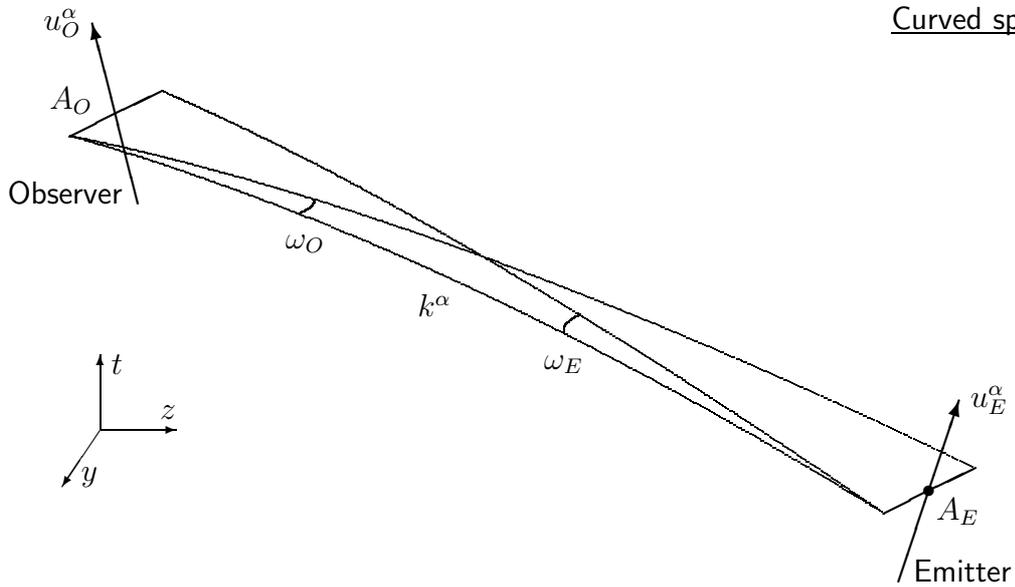
Flat space



For no relative motion, both distances are the same;

$$\frac{A_O}{\omega_E} = d^2 = \frac{A_E}{\omega_O}$$

Curved space-time



For small solid angles, both the converging and diverging rays encounter the same Riemann tensor along their paths, but relative motion (and hence redshift) affects areas, angles, frequencies, and photon arrival rates;

$$A_O \omega_O (k^\alpha u_\alpha)_O = A_E \omega_E (k^\alpha u_\alpha)_E$$

$$\rightarrow d_D^2 = \frac{d_L^2}{(1+z)^4}$$

## 7 FLRW Observational Relations

As an illustration, the standard expected FLRW relations are

Metric:

$$ds^2 = -dt^2 + S^2 \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right\}$$

Past null cone:

$$\int \frac{dt}{S(t)} = \int \frac{-dr}{\sqrt{1 - kr^2}} \rightarrow \hat{t}(r)$$

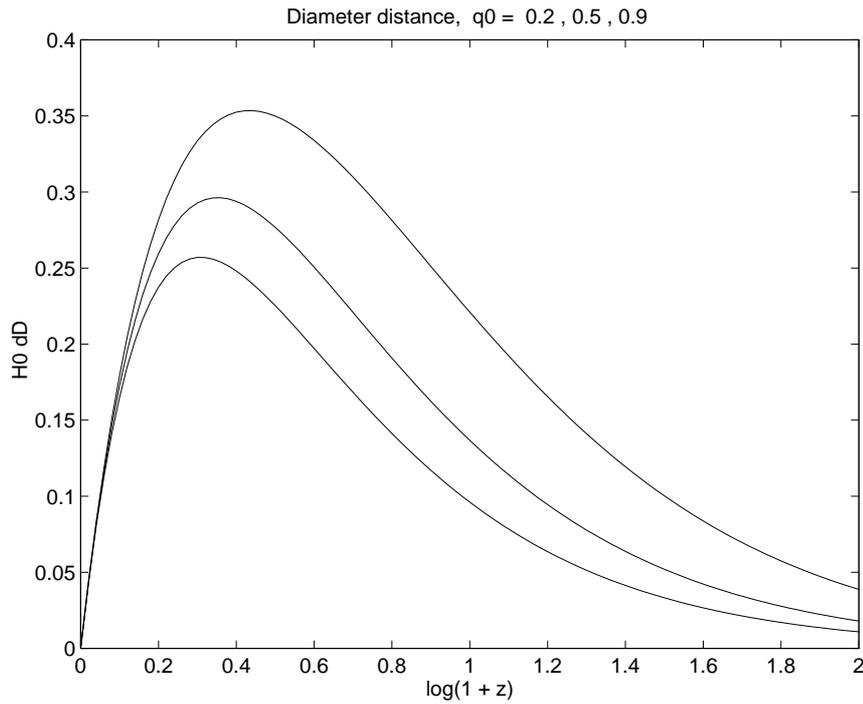
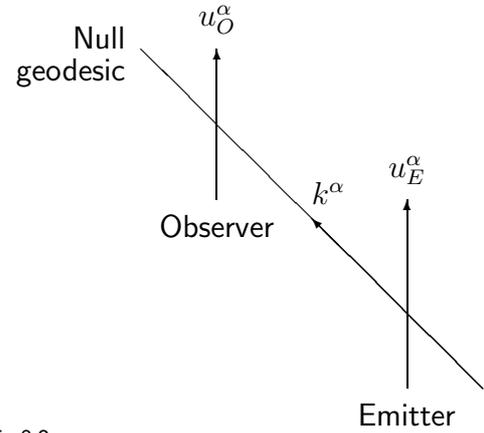
Redshift:

$$z = \frac{\Delta\lambda}{\lambda_E}$$

$$1 + z = \frac{\lambda_O}{\lambda_E} = \frac{(k^\alpha u_\alpha)_E}{(k^\alpha u_\alpha)_O} = \frac{S_O}{S_E}$$

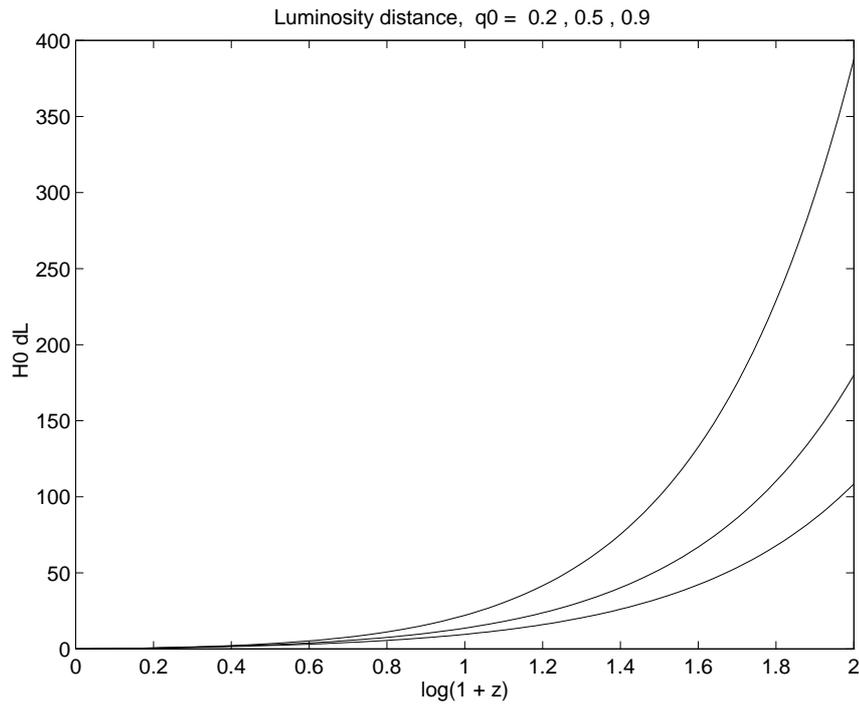
Diameter distance,  $d_D$  (take zero pressure  $p = 0$ ):

$$\frac{D}{\delta} = d_D = \frac{q_0 z + (1 - q_0)(1 - \sqrt{2q_0 z + 1})}{H_0 q_0^2 (1 + z)^2}$$



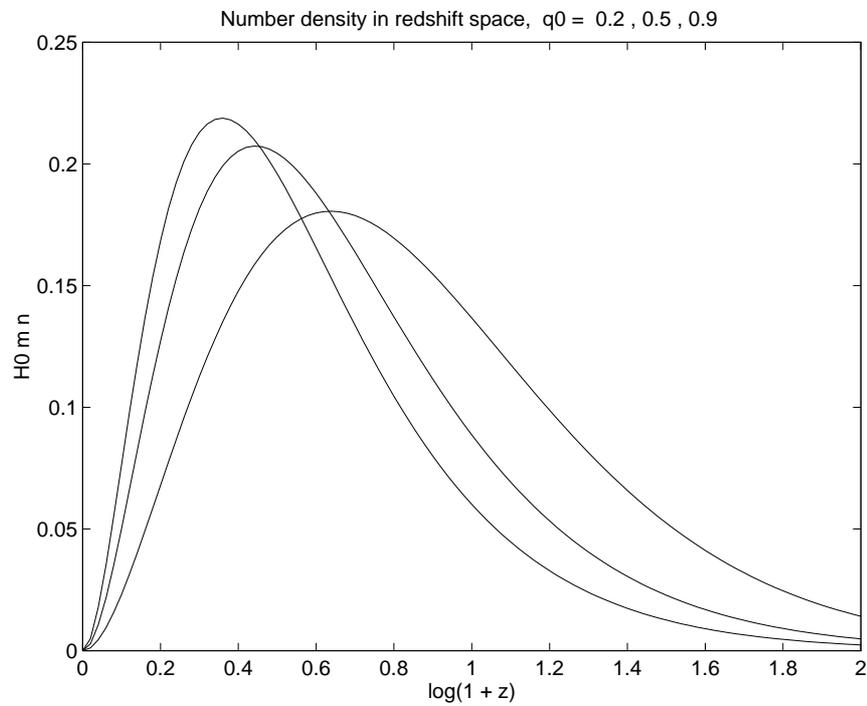
Luminosity distance,  $d_L$ :

$$\sqrt{\frac{L}{\ell}} = d_L = d_D(1+z)^2$$



Density in redshift space,  $n$ :

$$4\pi mn = \frac{3[q_0 z + 1 - q_0](1 - \sqrt{2q_0 z + 1})^2}{H_0 q_0^3 (1+z)^3 \sqrt{2q_0 z + 1}}$$



None of these observable!

Source evolution functions  $D, m, L$  needed.

## 8 Currently Prevailing Opinion

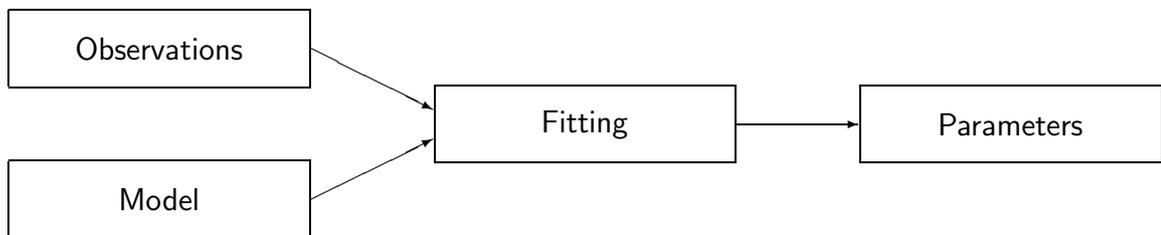
- No source evolution for SN
- Homogeneity guaranteed by Copernical Principle
- Therefore add  $\Lambda$  (adjust eq of state) to fit observations.

BUT

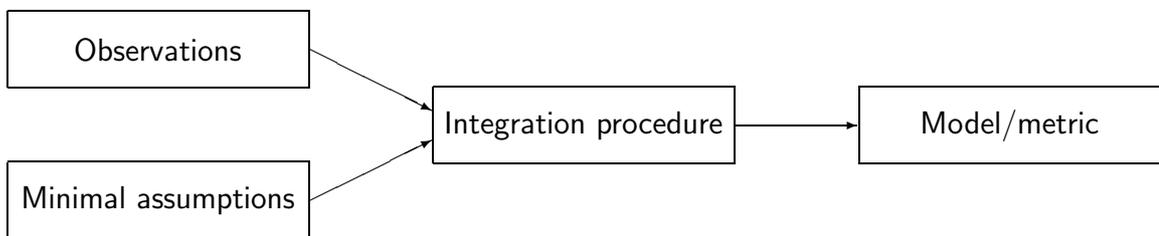
- Can't we prove homogeneity?
- If we assume homogeneity, there's a danger of a circular argument.

## 9 Metric of the Cosmos

Rather than fitting observations to a chosen model,



instead let observations dictate the model.



# 10 A Selection of Papers on Relating Observations to the Metric in Inhomogeneous Cosmologies

[Not representative]

1. I.M.H. Etherington (1933)  
 "On the Definition of Distance in General Relativity"  
*Phil. Mag. VII* **15**, 761-73 (1933)
  - Criticises luminosity distance definitions of Whittaker & Tolman, and proposes own.
  - First to obtain the reciprocity theorem;
  - Applies to de Sitter metric.
  
2. G. Temple  
 "New Systems of Normal Coordinates for Relativistic Optics"  
*Proc. Roy. Soc. London A*, **168**, 122-48 (1938)
  - Introduces observational coordinates (see below) — "optical coords"
  
3. W.H. McCrea  
 "Observable Relations in Relativistic Cosmology. II"  
*Zeitschrift für Astrophysik* **18**, 98 (1939)  
 reprinted in *Gen. Rel. Grav.* **30**, 315-33 (1998)
  - Observational relations in an inhomogeneous metric;
  - Applies EFEs for comoving dust;
  - Second order series approx, in affine parameter  $\lambda$  along null cone, for
    - $z$ , including a mean relation between  $z$ ,  $\rho$ ,  $\lambda$ ,
    - $d_L$  (coeff expressions incomplete);
  - Application to homogeneous case.
  
4. J. Kristian and R.K. Sachs  
 "Observations in Cosmology"  
*Astrophys. J.* **143**, 379-99 (1966)
  - Define corrected luminosity distance
 
$$d_{LC} = \sqrt{\frac{L_E}{L_O}} \left( \frac{\nu_O}{\nu_E} \right)^2 = d_D$$
  - Series expansions near observer in a general metric for
    - $1 + z$  in terms of  $d_{LC}$  & source motion,
    - reciprocity theorem,
    - image distortion by non-uniform geometry,
    - $n \frac{dz}{dd_{LC}}$  in terms of  $d_{LC}$  & density field,
    - proper motion;
  - Case of dust;
  - Parameter estimates/limits from observations
    - homogeneity not proven;

- Source evolution barely mentioned.

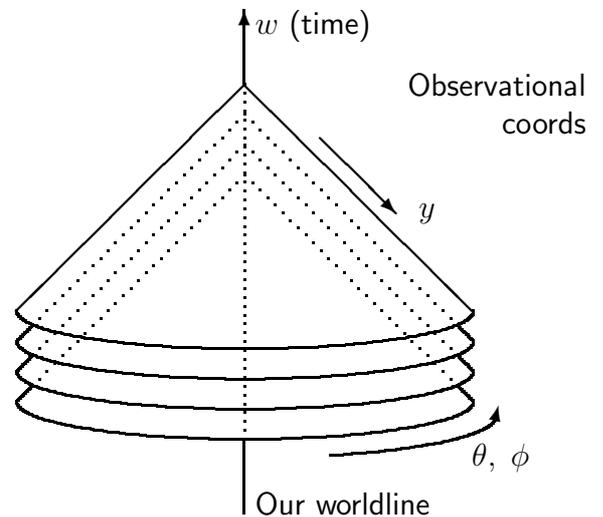
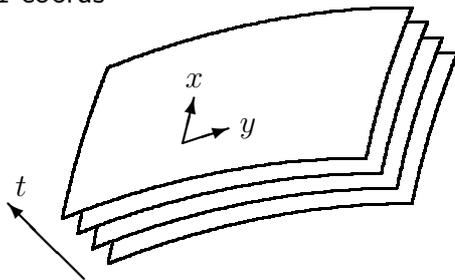
5. G.F.R. Ellis  
 "Relativistic Cosmology"  
 pp 104-79 of  
*General Relativity and Cosmology*  
 XLVII Enrico Fermi Summer School (Varenna)  
 Ed. R. K. Sachs (Academic Press, 1971)

- Includes excellent review & derivation of observational relations (§6);
- Mostly in spherical symmetry;
- General form of reciprocity theorem.

6. G.F.R. Ellis, S.D. Nel, R. Maartens, W.R. Stoeger, & A.P. Whitman  
 "Ideal Observational Cosmology"  
*Phys. Reports* **124**, 315-417 (1985)

- Major review
- Observational coordinates

Theoretician's  
3+1 coords



- general metric form:

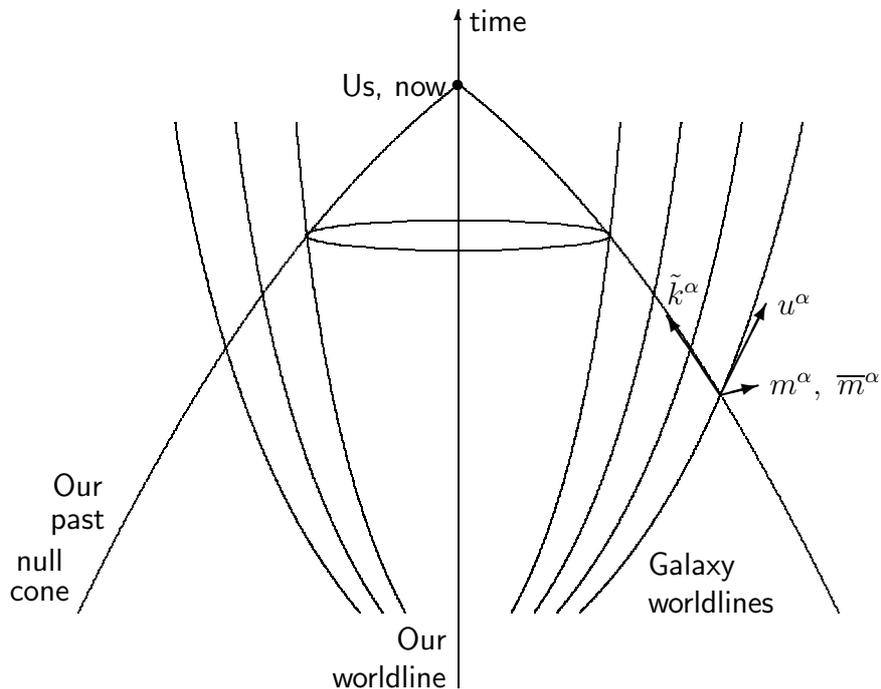
$$g_{ab} = \begin{pmatrix} \alpha & \beta & v_2 & v_3 \\ \beta & 0 & 0 & 0 \\ v_2 & 0 & h_{22} & h_{23} \\ v_3 & 0 & h_{23} & h_{33} \end{pmatrix}$$

- central limits;

- Observational relations
  - functions of  $z$
  - in these coords
- Example: isotropic observations + spherically symmetric metric;
- Cosmography (metric but no field equations):
  - standard observations don't determine even our past light cone fully,
  - can't prove spherical symmetry from isotropic observations,
  - can't prove homogeneity;
- Problem of source evolution noted;

- Cosmology — use EFEs too:
  - use NP formalism,
  - assume source evolution known,
  - give solution method for dust,
  - obtain central limits,
  - prove existence & uniqueness of solutions;
- Power series approximations;
- Example — spherical symmetry & FLRW.

7. W.R. Stoeger, S.D. Nel, R. Maartens & G.F.R. Ellis  
 “The Fluid-Ray Tetrad Formulation of Einstein’s Field Equations”  
*Class. Q. Grav.*, **9**, 493-507 (1992),  
 R. Maartens, Ph.D. thesis (1980)



- Fluid-ray (FR) tetrad  $\{e_a\} = \{u, \tilde{k}, m, \bar{m}\}$  where
  - $\tilde{k}^\alpha =$  null cone tangent vector
  - $u^\alpha =$  fluid 4-velocity
  - $m^\alpha, \bar{m}^\alpha =$  angular basis vectors;
- Developed in analogy with NP tetrad;

## 8. Other papers in the “Observational Cosmology Programme”

W.R. Stoeger, G.F.R. Ellis & S.D. Nel

“Observational Cosmology: III. Exact Spherically Symmetric Dust Solutions”  
*Class. Q. Grav.*, **9**, 509-26 (1992a).

W.R. Stoeger, S.D. Nel & G.F.R. Ellis

“Observational Cosmology: IV. Perturbed Spherically Symmetric Dust Solutions”  
*Class. Q. Grav.*, **9**, 1711-23 (1992b).

W.R. Stoeger, S.D. Nel & G.F.R. Ellis

“Observational Cosmology: V. Solutions of the First Order General Perturbation Equations”  
*Class. Q. Grav.*, **9**, 1725-51 (1992c).

R. Maartens, D.R. Matravers  
“Isotropic and Semi-Isotropic Observation in Cosmology”  
*Class. Q. Grav.* **11**, 2693-704 (1994).

M.E. Araújo & W.R. Stoeger  
“Exact Spherically Symmetric Dust Solution of the Field Equations in Observational Coordinates with Cosmological Data Functions”  
*Phys. Rev. D*, **60**, 104020, 1-7 (1999); plus Errata in *Phys. Rev. D*, **64**, 049901, 1 (2001).

M.E. Araújo, R.C. Arcuri, J.L. Bedran, L.R. de Freitas, & W.R. Stoeger  
“Integrating Einstein Field Equations in Observational Coordinates with Cosmological Data Functions: Nonflat Friedmann-Lemaitre-Robertson-Walker Cases”  
*Astrophys. J.* **549**, 716-20 (2001).

M.E. Araújo, S.R.M.M. Roveda & W.R. Stoeger  
“Perturbed Spherically Symmetric Dust Solution of the Field Equations in Observational Coordinates with Cosmological Data Functions”  
*Astrophys. J.* **560**, 7-14 (2001).

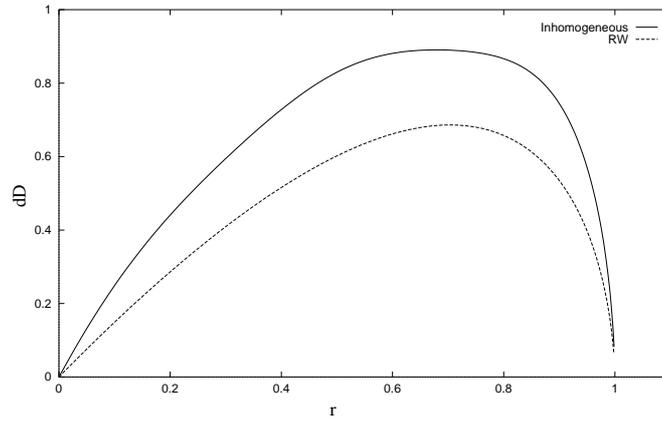
9. W. Rindler & D. Suson  
“How to Determine a Tolman-Bondi Universe from Ideal Observable and Theoretical Relations”  
*Astron. & Astrophys.* **218**, 15-8 (1989)

- “Proper” treatment of source evolution;
- In-principle solution only:
  - one can find DEs depending on ..... ,
  - if this is known, one could solve for that;
- Intrinsically inhomogeneous model.
- Suggested obtaining the source evolution from discrepancies between observations and those expected in homogeneous models.

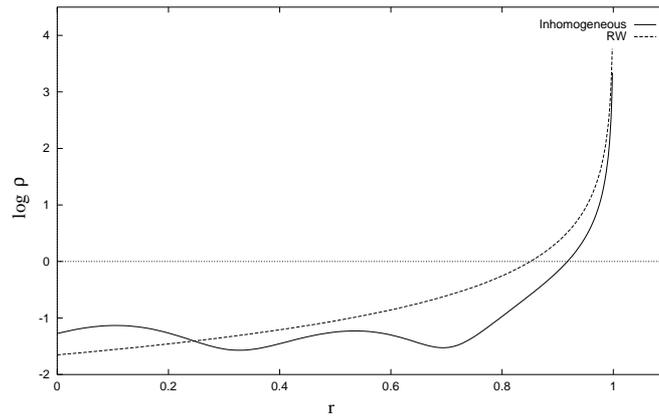
10. N. Mustapha, B.A.C.C. Bassett, C. Hellaby and G.F.R. Ellis  
“The Distortion of the Area Distance-Redshift Relation in Inhomogeneous Isotropic Universes”  
*Class. Q. Grav.* **15**, 2363-79 (1997)

- Showed inhomogeneous models don't necessarily average out to FLRW observational relations.
- Showed how inhomogeneity distorts observational relations
  - redshift may not be a monotonic function of distance away, creating loops in the observational plots.

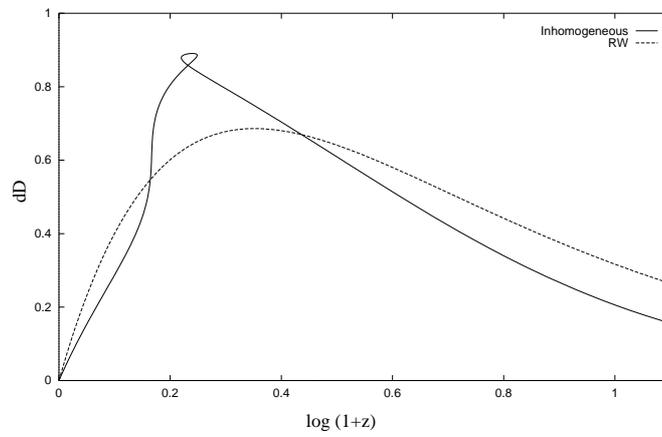
Diameter distance as a function of radial coordinate:



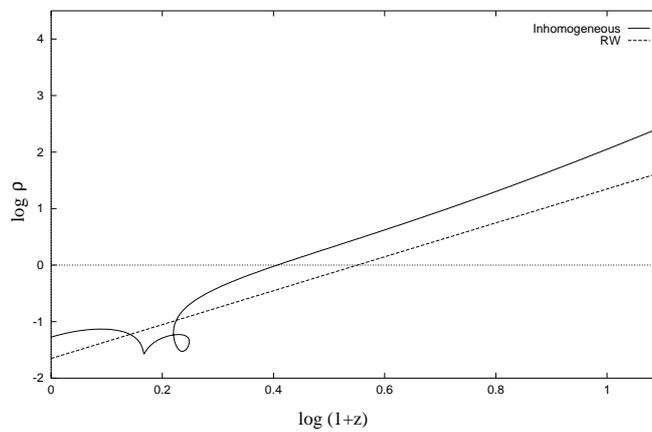
Diameter distance as a function of redshift:



Density as a function of radial coordinate:



Density as a function of redshift:



11. N. Mustapha, C. Hellaby and G.F.R. Ellis  
 "Large Scale Inhomogeneity Versus Source Evolution: Can We Distinguish Them Observationally?"  
*Mon. Not. Roy. Astron. Soc.* **292**, 817-30 (1997)

- Theore: given any (reasonable) observations

$$\delta(z), \quad \ell(z), \quad n(z)$$

and any (reasonable) source evolution functions

$$D(z), \quad L(z), \quad m(z)$$

you can find an Lemaître-Tolman model that fits it.

- Gave detailed DEs + algorithm + existence.
- Special cases of theorem:
  - any observations + zero source evolution  $\rightarrow$  LT,
  - any observations + suitable evolutions functions  $\rightarrow$  FLRW;

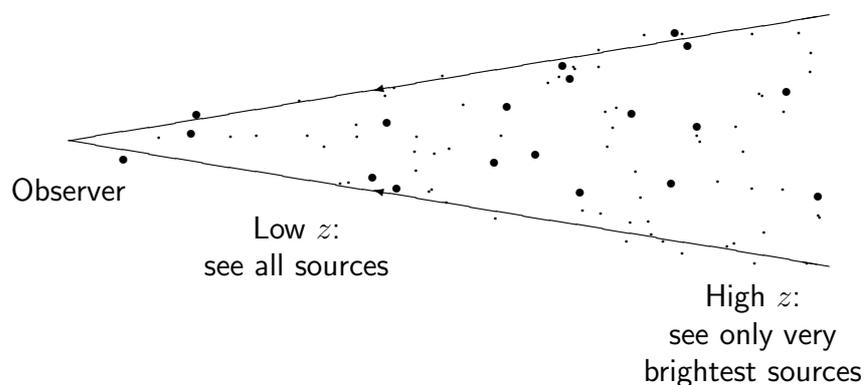
12. C. Hellaby  
 "Multicolour Observations, Inhomogeneity and Evolution"  
*Astron. Astrophys.* **372**, 357-63 (2001)

- Multicolour observations can be used to test source evolution theories, independently of the model;
- Basic point — all frequencies suffer same gravitational/geometric effects.

13. M.B. Ribeiro & W.R. Stoeger  
 "Relativistic Cosmology Number Counts and the Luminosity Function"  
*Astrophys. J.* **592**, 1-16 (2003)

- Brings cosmology of theoreticians & observers closer;
- Focusses on luminosity function:

$$\phi(L) = \text{relative frequency of galaxies with absolute luminosity } L$$



- estimate  $\phi$  at low  $z$ , where all galaxies are seen,
- extrapolate it to high  $z$ , where only very brightest are seen,
- use to estimate true numbers.
- At low  $L$ ,  $\phi$  uncertain — is there a cut-off (usually assumed) or lots of really faint galaxies?

- Used CNOC2 data (200 galaxies,  $0.12 < z < 0.55$ , and 3 “morphological types” observed in 3 colour bands);
- Used Schechter luminosity function

$$\phi = \phi_* \mathcal{L}^\alpha e^{-\mathcal{L}}, \quad \mathcal{L} = \frac{L}{L_*}$$

where constants  $\phi_*$ ,  $L_*$ ,  $\alpha$  are determined observationally, and  $z$  variation (source evolution) functions  $\phi_*(z)$ ,  $L_*(z)$  were assumed (with zero evolution for  $\alpha$ ).

- They checked for consistency with Einstein de Sitter model (i.e. checked original data reduction):
  - found general consistency,
  - some deviation at higher  $z$ .

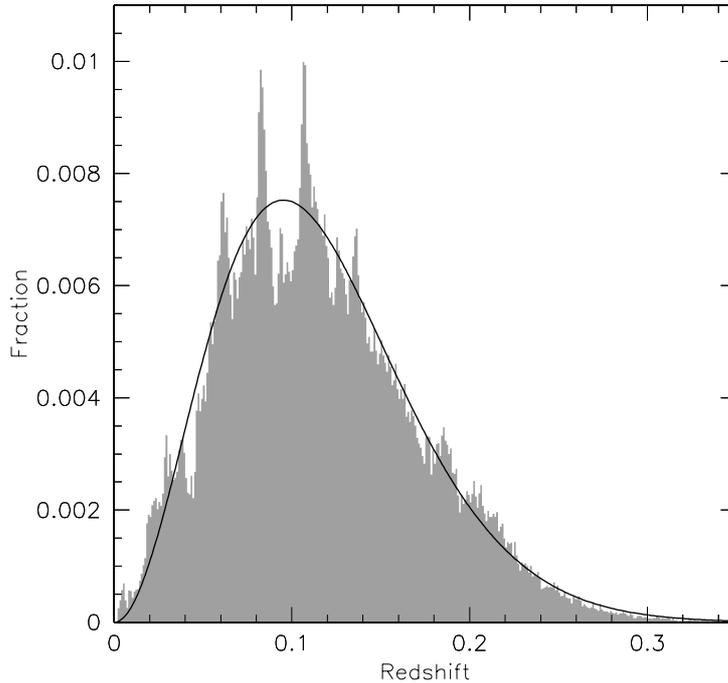
## 11 Towards an Implementation of the MHE algorithm

- Re-write the integrals as a system of ODEs, to find the 3 arbitrary functions,  $M(r)$ ,  $E(r)$  &  $t_B(r)$ , that define an LT model:

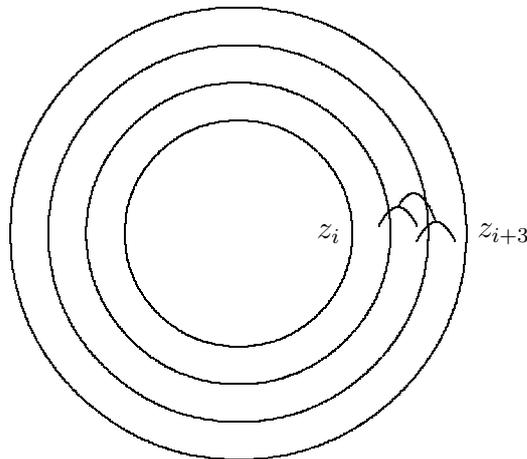
$$\begin{aligned} \frac{dr}{dz} &= \frac{1}{\zeta} \\ \frac{d\zeta}{dr} &= \frac{-4\pi mn\zeta(1+z) - \left[ \frac{d^2 d_D}{dz^2}(1+z) + \frac{dd_D}{dz} \right] \zeta^2}{\frac{dd_D}{dz}(1+z)} \\ \frac{dM}{dr} &= \frac{2\pi mn \left[ \left( \zeta \frac{dd_D}{dz} \right)^2 + 1 - \frac{2M}{d_D} \right]}{\frac{dd_D}{dz}} \\ E &= \text{algebraic eq} \\ t_B &= \text{algebraic eq} \end{aligned}$$

- Once  $M(r)$ ,  $E(r)$  &  $t_B(r)$  are known, easy to reconstruct model evolution.
- Real data is discrete
  - many “point” sources — not continuous density,
  - to plot, group into  $dz$  bins.

The relative frequency of sources in the 2dF Galaxy survey, arranged in redshift bins. From M. Colless et al, *Mon. Not. Roy. Astron. Soc.* **328** 1039 (2001), "The 2dF Galaxy Redshift Survey: Spectra and Redshifts". [The drop in numbers above  $z = 0.1$  is because the sources are too faint to be detected. The intrinsic maximum in the  $n(z)$  plot should be at much larger  $z$ , around  $z = 1.6$ .]



- Either fit smooth curve to data (have to assume functional form)
  - or discretise ODEs (assumes less)
- Need 2nd derivative of  $d_D$  — takes 3  $dz$  bins — first value of  $\Delta\Delta d_D$  out from origin.



- No data at origin, but light rays end at the origin — assume an extension.
- Need to test numerics — create an artificial universe and generate "observations" for which "correct" metric known.

## 12 Conclusions

- We need to gain experience in data reduction by starting with very simple cases, and gradually include more effects, such as angular variation, lensing etc.
- We must continue the theory-observation rapprochement.
- Be ready for the expected flood of data.
- Knowing the metric nearby will assist in analysing distant observations in more than just a statistical sense.
- Eventually we'll be able to quantify the degree of homogeneity.

## 13 Acknowledgements

I would like to thank the organisers Mathematics of Gravitation for their invitation to speak, and for their generous hospitality. I also thank the South African NRF for a research grant.

## A Other Relevant Papers

G. Lemaître (1933) *Ann. Soc. Scient. Bruxelles* **A53**, 51-85, "L'Univers en Expansion"; reprinted in (1997) *Gen. Rel. Grav.* **29**, 641-80.

R.C. Tolman (1934) *Proc. Nat. Acad. Sci.* **20**, 169-76, "Effect of Inhomogeneity on Cosmological Models"; reprinted in (1997) *Gen. Rel. Grav.* **29**, 935-43.

H. Bondi (1947) *Mon. Not. Roy. Astron. Soc.* **107**, 410-25, "Spherically Symmetric Models in General Relativity"; reprinted in (1999) *Gen. Rel. Grav.* **31**, 1783-805.

J. Ehlers, P. Geren & R.K. Sachs (1968), *J. Math. Phys.*, **9**, 1344-9, "Isotropic Solutions of the Einstein-Liouville Equations".

G.F.R. Ellis (1975) *Qu. J. Roy. Astron. Soc.* **16**, 245-64, "Cosmology and Verifiability".

W.R. Stoeger (1987) in *Theory and Observational Limits in Cosmology*, ed. W. R. Stoeger (Vatican City : Vatican Obs.), 287.

G.F.R. Ellis & W.R. Stoeger (1987) *Class. Q. Grav.*, **4**, 1697-729, "The 'Fitting Problem' in Cosmology".

W.R. Stoeger, R. Maartens, & G.F.R. Ellis (1995), *Astrophys. J.*, **443**, 1, "Proving Almost Homogeneity of the Universe: An Almost Ehlers-Geren- Sachs Theorem".

R. Maartens, G. F. R. Ellis, & W. R. Stoeger (1995a) *Phys. Rev. D* **51** 1525-35, "Limits on Anisotropy and Inhomogeneity from the Cosmic Background Radiation".

R. Maartens, G. F. R. Ellis, & W. R. Stoeger (1995b) *Phys. Rev. D* **51** 5942-5, "Improved Limits on Anisotropy and Inhomogeneity from the Cosmic Background Radiation".

R. Maartens, G. F. R. Ellis, & W. R. Stoeger (1996) *Astron. Astrophys.* **309** L7-10, "Anisotropy and Inhomogeneity of the Universe from  $\Delta T/T$ ".

R. Maartens, N.P. Humphreys, D.R. Matravers, & W.R. Stoeger (1996) *Class. Q. Grav.*, **13**, 253-64. "Inhomogeneous Universes in Observational Coordinates"; plus Errata in (1996) *Class. Q. Grav.*, **13**, 1689-90.

W.R. Stoeger, M.E. Araujo & T. Gebbie (1997) *Astrophys. J.* **476** 435-9, "The Limits on Cosmological Anisotropies and Inhomogeneities from COBE Data"; Errata: (1999) *Astrophys. J.* **522**, 559.

M.N. Celerier, (2000) *Astron. Astrophys.* **353** 63-71, "Do We Really See a Cosmological Constant in the Supernovae Data?"