

Shell Crossings and the Tolman Model¹

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Abstract

We consider the problem of shell crossings and regular maxima in the Tolman model. The necessary and sufficient conditions which guarantee no shell crossings will arise in Tolman Models are derived, and we show explicitly that a Tolman model (in general, with a surface layer) may contain both elliptic and hyperbolic regions without developing any shell crossings and without the hyperbolic regions recollapsing. This finding is contrary to the recent hypothesis of Zel'dovich and Grishchuk. We also show that the properties that distinguish shell crossings from more serious singularities in spherical symmetry are independent of the equation of state.

Subject heading: Cosmology

I Introduction

The Tolman model, which describes comoving spherically symmetric dust, has been widely used to study inhomogeneous cosmologies and gravitational collapse. One of the problems with using this model, however, is that shell crossings can easily develop, and the spacetime cannot be continued through them, since the Kretschmann scalar diverges, and there is not a unique continuation (e.g., Seifert 1979). If one treats the shell crossing itself as a boundary surface within the Tolman model, the region beyond is unacceptable, since it has negative density. (Although the procedure of swapping particle labels at the shell crossing may remove the problem, it also violates the comoving assumption, and we do not accept it as valid.) It is therefore of interest to find the minimum conditions which guarantee no shell crossings will occur. The conditions which are usually given to avoid the formation of shell crossings (e.g. Hellaby and Lake 1984; Landau and Lifshitz 1975) are, in fact, too restrictive.

The problem of shell crossings is often avoided by taking them to be part of the big bang or recollapse surfaces; however, in the Tolman model there are some important distinctions between shell crossings, where only one metric component goes to zero (g_{RR}), and the true big bang and recollapse surfaces, where where the angular metric components go to zero ($g_{\theta\theta}$ and $g_{\phi\phi}$).

(i) The shell crossing surfaces are always timelike, while the bang and recollapse surfaces are always spacelike.

(ii) The frequency shift of light coming from the shell crossing surfaces is finite (red or blue), while

¹Post-publication: The published errata have been incorporated in the main text. (Bill Bonnor had pointed out that — in the uncorrected, published version — the derivation in §IIb*i* was flawed, and the model given in §IIIc was not fully regular. There was also a minor error in equation (A15).) Otherwise differences from the published version are purely stylistic.

that from the bang surface is an infinite redshift, except along radial directions which, in general, display an infinite blueshift (but see Hellaby and Lake 1984 for further details).

(iii) The surface density, σ , on shell crossings remains finite (Lake 1984a), while on the bang and recollapse surfaces, σ diverges. In the Appendix we show that these properties of shell crossings hold in all spherically symmetric models regardless of the equation of state. (Müller zum Hagen, Yodzis, and Seifert 1974 have shown that shell crossings can indeed occur in models which contain pressure.)

For these reasons we believe that shell crossings are not serious physical singularities, but rather they indicate the breakdown of the basic assumptions of the Tolman metric. These assumptions are that the matter can be represented by comoving coordinates and a single-particle four-velocity at each point. Although there may well be a way of resolving this problem, it cannot be dealt with in the context of the Tolman model. It is worth noting that the theorems which use the Raychaudhuri equation (e.g. Landau and Lifshitz 1975) to predict a divergence in the density fail to distinguish shell crossings from more serious physical singularities. As Seifert has pointed out, even when the origin of a singularity is really hydrodynamic, the Einstein equations ensure that a curvature singularity also appears.

In the next section we derive the necessary and sufficient conditions for no shell crossings in Tolman models, and we apply them in §III, where we present a closed hyperbolic model, an open elliptic model, and a closed hybrid model. The third model provides a counterexample to Zel'dovich and Grishchuk's (1984) hypothesis that such a model must completely recollapse. Section IV is the discussion.

II Conditions for No Shell Crossings

a) Review of the Basic Equations

We use the notation of Zel'dovich and Grishchuk (1984). The Tolman metric is (Lemaitre 1933; Tolman 1934; Bondi 1947)

$$ds^2 = -dt^2 + \frac{r'^2}{1+f} dR^2 + r^2 d\Omega^2, \quad (2.1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, the single prime denotes $\partial/\partial R$, the dot denotes $\partial/\partial t$, and the evolution of $r(R, t)$ is given by the following parametric equations.

$f > 0$,

$$r = \frac{F}{2f}(\cosh \eta - 1), \quad (\sinh \eta - \eta) = \frac{2f^{3/2}(t - a)}{F}; \quad (2.2)$$

$f = 0$,

$$r = \left[\frac{9F(t - a)^2}{F} \right]^{1/3}; \quad (2.3)$$

$f < 0$,

$$r = \frac{F}{2(-f)}(1 - \cos \eta), \quad (\eta - \sin \eta) = \frac{2(-f)^{3/2}(t - a)}{F}. \quad (2.4)$$

The functions $a(R)$, $f(R)$, and $F(R)$ are all arbitrary functions of the radial coordinate R , which allow a coordinate choice plus the specification of two physically independent quantities. The local time at which $r = 0$ is $a(R)$, and in the region $t \geq a$, it is the big bang time, while for $t \leq a$ it is the recollapse time. The function $F(R)$ is proportional to the effective gravitational mass within radius R (e.g., Bondi 1947), and $f(R)$ determines both the type of time evolution and the local geometry. We can define $\tilde{\omega}(R)$ — a local value of π — in terms of the rate of change of areal radius with respect to proper radius, on some constant time slice, with

$$\tilde{\omega}(R) = \frac{\pi(\partial_R \sqrt{g_{\theta\theta}})^2}{g_{RR}} = \pi[1 + f(R)] .$$

Also $f(R)$ may be regarded as a local energy constant. For hyperbolic and parabolic regions, the expansion rate at late times is given by $\dot{r} = f^{1/2}$, while in the elliptic regions, the “mean velocity”, defined by dividing radius, r_{\max} , at maximum expansion by the time from bang to r_{\max} , is just $2(-f)^{1/2}/\pi$. Since F is proportional to the mass, it must be everywhere positive, and $f \geq -1$ must hold for a Lorentzian manifold. The density is given by

$$8\pi\rho = \frac{F'}{r'r^2} , \quad (2.5)$$

and r' can be given, for all f , as

$$r' = \left(\frac{F'}{F} - \frac{f'}{f} \right) r - \left[a' + \left(\frac{F'}{F} - \frac{3f'}{2f} \right) (t - a) \right] \dot{r} , \quad (2.6)$$

where

$$\dot{r}^2 = \frac{F}{r} + f .$$

An origin occurs at R_0 if $r(R_0, t) = 0$ for all t . Assuming a remains finite, this requires that, near the origin, $f \propto F^s$, where $s > 0$ and $F \rightarrow 0$ at R_0 . We call the origin regular if $s = \frac{2}{3}$, since the time evolution is then the same as the surrounding regions, and the density is finite there.

For the purpose of this paper, we define the terms ‘elliptic,’ ‘parabolic,’ and ‘hyperbolic’ to signify only the local type of time evolution (i.e., $f < 0$, $f = 0$, $f > 0$, respectively). The terms “open” and “closed” refer only to topological properties of the model, so that, for spherically symmetric metrics, closed models have two values of R where $g_{\theta\theta} = 0$, while open models have only one (or conceivably none).

b) Shell Crossings

In what follows, we shall consider only the case $t \geq a$. For $t \leq a$ one merely has to replace $t - a$ with $a - t$, and a' with $-a'$. We define shell crossings to be surfaces on which $r' = 0$ and where the density, ρ , diverges.

A regular extremum (or point of inflection) in r along constant time slices may occur without causing a shell crossing, provided ρ does not diverge, as was made clear by Zel'dovich and Grishchuk (1984). By equation (2.5), this implies

$$F' = 0 \quad (2.7)$$

wherever $r' = 0$, and also that the surface $r' = 0$ remains at fixed R , say R_m . Consider equation (2.6) at R_m . Since the coefficients of a' and f' are different functions of time, then

$$a' = 0 \quad (2.8)$$

and

$$f' = 0 \quad (2.9)$$

must also obtain at R_m . Thus, the condition for a regular maximum in $r(R, t)$ is that equations (2.7), (2.8), and (2.9) all hold at the same R . In fact, the extrinsic curvature, K_{ij} (defined in the Appendix), shows a jump in its $\theta\theta$ component at R_m , so there is a surface layer of mass $M = 2r(1 + f)^{1/2}$. Despite this, F , ρ , and $g_{\alpha\beta}$ are all continuous and finite through the layer.

i) Elliptic Regions, $f < 0$

We next derive the conditions for no shell crossings in elliptic regions, $f < 0$. We define the functions $p(R)$ and $q(R)$ by

$$p = \frac{F}{(-f)}, \quad (2.10)$$

and

$$q = \frac{F}{(-f)^{3/2}}. \quad (2.11)$$

The value of $r(R, t)$ at maximum expansion ($\eta = \pi$) is just $p(R)$, while the time from bang to recollapse along constant R is $\pi q(R)$. Clearly, both p and q are positive everywhere; i.e.,

$$p \geq 0 \quad \text{and} \quad q \geq 0. \quad (2.12)$$

We may write the radial derivative of r as

$$\frac{r'}{r} = \frac{p'}{p} - \frac{q'}{q}\phi_1 - \frac{2a'}{q}\phi_2, \quad (2.13)$$

where

$$\phi_1(\eta) = \frac{\sin \eta(\eta - \sin \eta)}{(1 - \cos \eta)^2}, \quad (2.14)$$

and

$$\phi_2(\eta) = \frac{\sin \eta}{(1 - \cos \eta)^2}. \quad (2.15)$$

The functions $\phi_1(\eta)$ and $\phi_2(\eta)$ are shown in Figure 1.

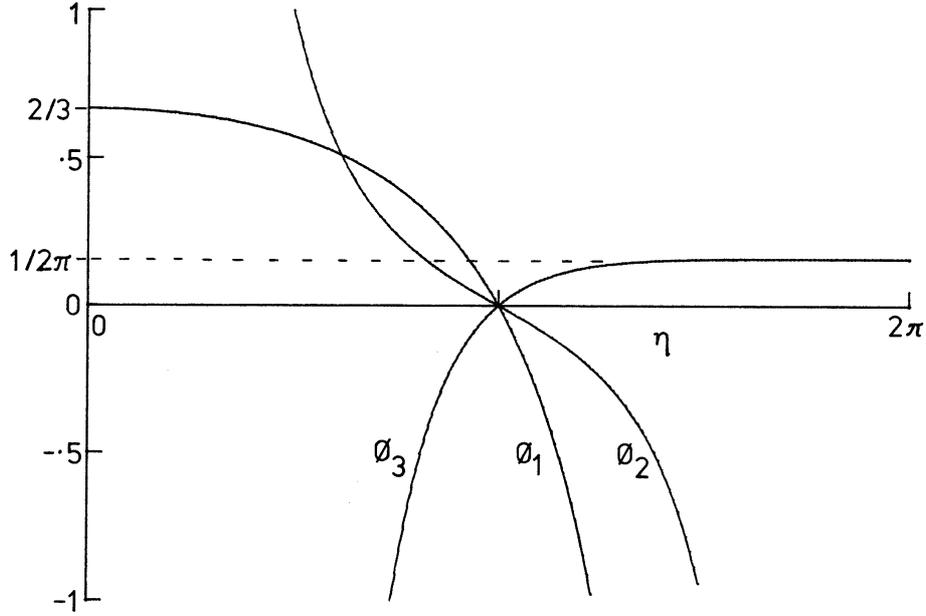


Fig. 1. The behavior of the functions $\phi_1(\eta)$, $\phi_2(\eta)$ and $\phi_3(\eta, \alpha = \frac{2}{3})$, defined by eqs. (2.14), (2.15), and (2.21). For all values of $\alpha \geq \frac{2}{3}$, ϕ_3 is similar to the curve shown, and, in particular, the upper limit is always $1/2\pi$.

For $r' > 0$, it is clear from equation (2.5) that²

$$F' \geq 0 \quad (2.36)$$

is required for positive density. Consider the evolution of equation (2.13) with time, i.e., as η goes from 0 to 2π . At early times (small η) ϕ_2 dominates, going to $+\infty$, so we must have

$$a' \leq 0 \quad (2.37)$$

for $r' > 0$. At late times ($\eta \rightarrow 2\pi$), $\phi_2 \rightarrow -\infty$ and $\phi_1 \rightarrow 2\pi\phi_2$, meaning equation (2.13) becomes

$$\frac{r'}{r} = \left(-2\pi \frac{q'}{q} - \frac{2a'}{q} \right) \phi_1, \quad (2.38)$$

so that the third condition for $r' > 0$ is found to be

$$a' \geq \frac{-\pi F}{(-f)^{3/2}} \left(\frac{F'}{F} - \frac{3f'}{2f} \right). \quad (2.39)$$

Of course conditions (2.36), (2.37), and (2.39) also imply

$$\frac{F'}{F} \geq \frac{3f'}{2f}. \quad (2.40)$$

To demonstrate the sufficiency of these three conditions, the functions α , β , and ϕ_3 are defined by

$$\frac{p'}{p} = \alpha(R) \frac{q'}{q}, \quad (2.16)$$

²Post-publication: This section (IIIbi) is a merging of the original and the corrected argument given in the published erratum. Since there was no sensible correspondence between many of the equations in the two versions, distinct equation numbers were used in the erratum — (2.36)-(2.41) — and those numbers are retained here. This is why equation numbers are not sequential.

$$\frac{q'}{q} = -\beta(R) \frac{2a'}{q}, \quad (2.17)$$

and

$$\phi_3(\eta, \alpha) = \frac{-\phi_2}{(\alpha - \phi_1)}. \quad (2.21)$$

Examples of the function $\phi_3(\eta)$ are plotted in Figure 1 for $\alpha = \frac{2}{3}$, and in Figure 2 for $\alpha = 0.1, 0$, and -0.3 .

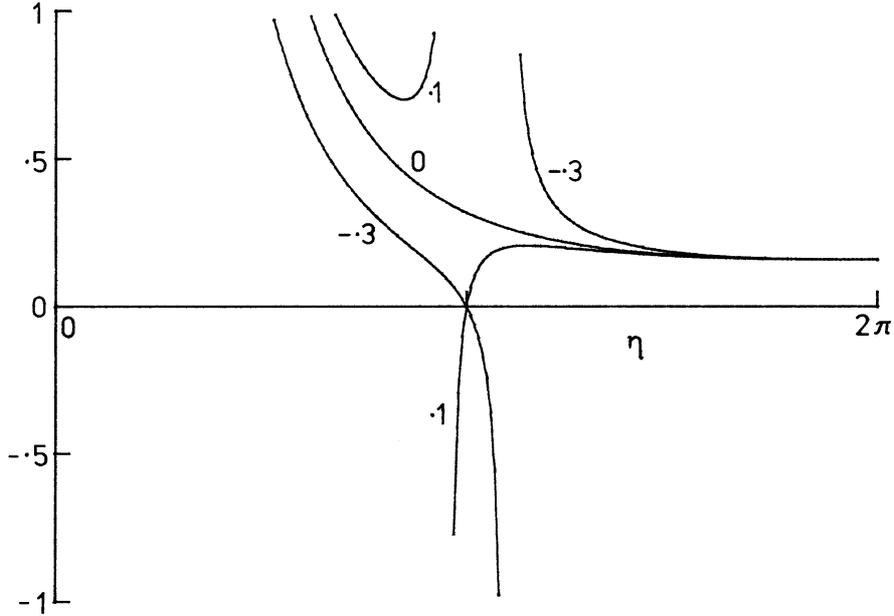


Fig. 2. The function $\phi_3(\eta, \alpha)$ has three possible forms other than the one shown in Fig. 1, but none of them have a finite upper limit. The sample curves shown here are labeled by their values of α .

For all $\alpha < \frac{2}{3}$, ϕ_3 has no upper limit, whereas for $\alpha \geq \frac{2}{3}$, ϕ_3 never exceeds $1/2\pi$ (at $\eta = 2\pi$). With these, equation (2.13) becomes

$$\frac{r'}{r} = -\frac{2a'}{q} [\beta(\alpha - \phi_1) + \phi_2]. \quad (2.18)$$

and conditions (2.36) and (2.39) become

$$\alpha \geq \frac{2}{3} \quad \text{and} \quad \beta \geq \frac{1}{2\pi}, \quad (2.41)$$

while equation (2.37) remains the same. It is obvious from equations (2.37), (2.41), and the figures that r' is always positive in equation (2.18). The converse of these conditions must hold for $r' < 0$. If both $F' = 0$ and $f' = 0$, then $a' = 0$ follows from equations (2.37) and (2.39), so $r' = 0$. Conditions (2.36) and (2.37) have an obvious meaning, and equation (2.39) ensures that the time of recollapse increases with R , wherever r' is positive. Thus equations (2.37) and (2.39) together ensure that $q' > 0$, and $p' > 0$, i.e. the time from bang to recollapse and the radius of maximum expansion both increase with r .

ii) **Hyperbolic Regions, $f > 0$**

The conditions for no shell crossings in hyperbolic regions, $f > 0$, are easier to derive. For this case the radial derivative is given by

$$\frac{r'}{r} = \frac{F'}{F}(1 - \phi_4) + \frac{f'}{f} \left(\frac{3}{2}\phi_4 - 1 \right) - \frac{2f^{3/2}a'}{F}\phi_5, \quad (2.26)$$

where

$$\phi_4 = \frac{\sinh \eta (\sinh \eta - \eta)}{(\cosh \eta - 1)^2}, \quad (2.27)$$

and

$$\phi_5 = \frac{\sinh \eta}{(\cosh \eta - 1)^2}. \quad (2.28)$$

The functions $\phi_4(\eta)$ and $\phi_5(\eta)$ are shown in Figure 3. Again consider the evolution of equation (2.26) as η goes from 0 to ∞ .

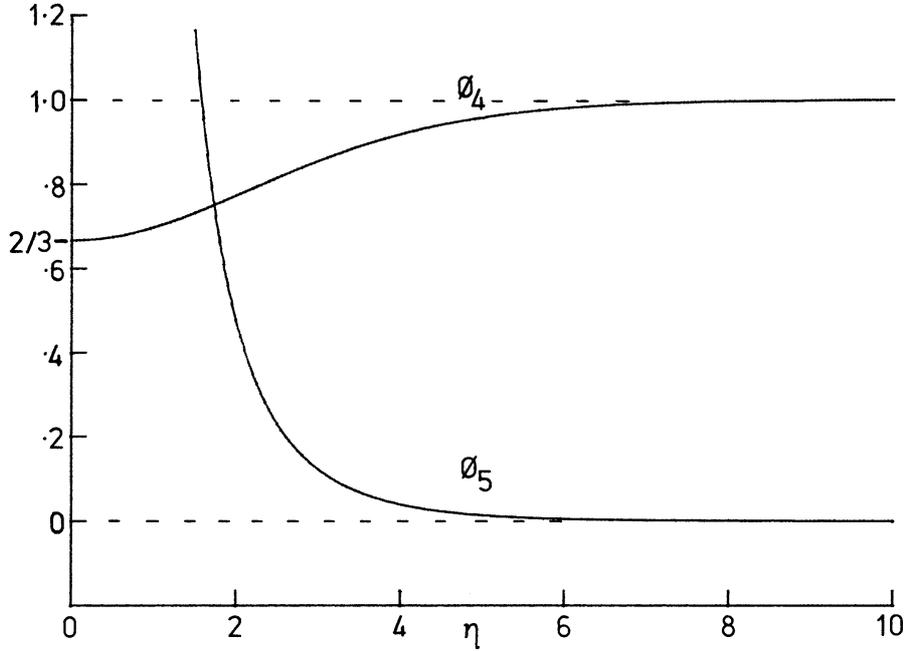


Fig. 3. The behavior of the functions $\phi_4(\eta)$ and $\phi_5(\eta)$, defined in eqs. (2.27) and (2.28).

At early times (small η), ϕ_5 dominates, going to $+\infty$, so we once more require

$$a' \leq 0 \quad (2.29)$$

for $r' > 0$. At late times $\phi_5 \rightarrow 0$, and $1 - \phi_4 \rightarrow 0$, so $3\phi_4/2 - 1$ dominates, and we must have

$$f' \geq 0. \quad (2.30)$$

We already know, from requiring the density to be non-negative, that

$$F' \geq 0. \quad (2.31)$$

Since $\frac{2}{3} \leq \phi_4 \leq 1$, it is obvious that conditions (2.29), (2.30), and (2.31) are sufficient as well as necessary. The converse of these conditions holds for $r' < 0$.

iii) Parabolic Regions, $f = 0$

The boundary between an elliptic and a hyperbolic region deserves special consideration, since the parameter η is not valid there. (The surfaces of constant η are independent on either side of $f = 0$, and they approach $t = \infty$ near the boundary.) An expansion of equations (2.2) and (2.4) in powers of f gives, after some manipulation,

$$\frac{r'}{r} = \frac{F'}{3F} - \frac{2a'}{3(t-a)} + \frac{3f'}{10} \left[\frac{2(t-a)^2}{3F^2} \right]^{1/3} + O(f), \quad (2.32)$$

which is valid for sufficiently small f in both $f > 0$ and $f < 0$ regions, and is exact for $f = 0$. From equation (2.32) it is evident that the conditions for no shell crossings are the same as for hyperbolic regions. For an extended parabolic region, $f' = 0$, equation (2.32) becomes the derivative of equation (2.3), but the same conditions obtain here also.

Lastly, we can verify that the density remains finite at a regular maximum, R_m , provided there are no shell crossings in the vicinity. Equation (2.6), with conditions (2.25) or (2.29)-(2.31), shows that r' cannot approach zero faster than F' does, so equation (2.5) must remain finite.

It is quite common in the literature (e.g. Hellaby and Lake 1984; Landau and Lifshitz 1975) to see the conditions for no shell crossings given as $r' > 0$, $F' > 0$. These are actually too restrictive, and they exclude regular maxima that must occur in closed models with well-behaved coordinate systems. (The usual Robertson-Walker coordinates, i.e. such that $g_{RR} = 1/[1 - kR^2]$, are defective at $R = R_m$.) As far as we know, Zel'dovich and Grishchuk are the first to have explicitly pointed out that, in a closed model, r' and F' must both be negative near one of the origins.

The conditions derived here, and summarised in Table 1, are not particularly restrictive; there is just one upper or lower bound on the gradient of each arbitrary function at each point.

$f \geq 0$	$f < 0$
$r' > 0$	
$a' \leq 0$	$a' \leq 0$
$f' \geq 0$	$a' \geq \frac{-\pi F}{(-f)^{3/2}} \left(\frac{F'}{F} - \frac{3f'}{2f} \right)$
$F' \geq 0$	$F' \geq 0$
but no more than two equalities at once	but not both $F' = 0$ and $f' = 0$ at once
$r' = 0$	
$a' = 0$	$a' = 0$
$f' = 0$	$f' = 0$
$F' = 0$	$F' = 0$
$r' < 0$	
$a' \geq 0$	$a' \geq 0$
$f' \leq 0$	$a' \leq \frac{-\pi F}{(-f)^{3/2}} \left(\frac{F'}{F} - \frac{3f'}{2f} \right)$
$F' \leq 0$	$F' \leq 0$
but no more than two equalities at once	but not both $F' = 0$ and $f' = 0$ at once

Table 1. Conditions for No Shell Crossings. These are the necessary and sufficient conditions in Tolman models which have $t \geq a$. The conditions for the case $t \leq a$ are found by replacing a' with $-a'$.

We give some examples of models without shell crossings in the next section.

III Some Examples

a) A Closed Hyperbolic Model

We choose the arbitrary Tolman functions to be

$$F = F_0 \gamma^m, \quad (3.1a)$$

$$f = f_0 \gamma^n, \quad (3.1b)$$

$$a = -a_0 \gamma^i, \quad (3.1c)$$

where $\gamma = \gamma(R)$ is a positive function of R that goes to zero at $R = 0$, and m , n , and i are all positive, as are F_0 , f_0 , and a_0 . Such models are hyperbolic, with no shell crossings, but regularity at the origin (as defined in §IIa) would require $2m = 3n$. If, for example, we set

$$\gamma(R) = 3 \sin\left(\frac{\pi R}{\ell}\right) + 2 \sin\left(\frac{3\pi R}{\ell}\right), \quad (3.2)$$

then, for fixed t , $r(R, t)$ has two maxima and one minimum, and it has a second origin at $R = \ell$, where $r(\ell, t) = 0$. In other words, the model is globally closed, though everywhere hyperbolic.

b) An Open Elliptic Model

For the second example, we make the choice³

$$F = \frac{F_0 R^m}{(1 + bR^n)}, \quad (3.7a)$$

$$f = \frac{-f_0 R^n}{(1 + bR^n)}, \quad (3.7b)$$

$$a = -a_0 R^i (1 + bR^n), \quad (3.7c)$$

where the constants F_0 , a_0 , f_0 , m , n , and i are all positive, and, as before, $2m = 3n$ for a regular origin. To ensure no shell crossings, we specify

$$2i = 2m - 3n, \quad (3.4a)$$

and

$$a_0 \leq \frac{\pi F_0}{f_0^{3/2}}. \quad (3.4b)$$

In this case there is only one origin, and the constant t , $\theta = \pi/2$ sections become conical at large R , so the space is elliptic, yet open.

³Post-publication: This equation triplet is a correction of the original and carries the equation number given in the published erratum.

c) A Closed Hybrid Model

For the final example, we specify the following functions,

$$F = F_0 R^3 (\ell - R)^3, \quad (3.5a)$$

$$f = f_0 (bR^3 - R^2) [b(\ell - R)^3 - (\ell - R)^2], \quad (3.5b)$$

$$a = a_0 (R^2 - \ell R), \quad (3.5c)$$

where a_0 , F_0 , f_0 , and ℓ are all greater than 0, and $b > 2/\ell$. The condition $f \geq -1$ puts an upper limit on f_0 in terms of b and ℓ , which is very long but not very instructive. It is clear that equations (3.5a)-(3.5c) cause no shell crossings in the hyperbolic region, and in the elliptic regions it is near the origins that the conditions for no shell crossings are hardest to satisfy. For small R , condition (2.25b) gives

$$\frac{(b\ell - 1)^{5/2}}{b^2} \leq \frac{3\pi F_0}{2a_0 f_0^{3/2}}. \quad (3.6)$$

The model then has no shell crossings and consists of a hyperbolic region between two elliptic regions, each of which contains a regular origin (at $R = 0$, and $R = \ell$). (There is no hyperbolic region if $b < 2/\ell$.) The evolution of the function $r(R, t)$ is shown in Figure 4 for $a_0 = 5$, $f_0 = 1$, $F_0 = 1$, $b = 3$, and $\ell = 1$.

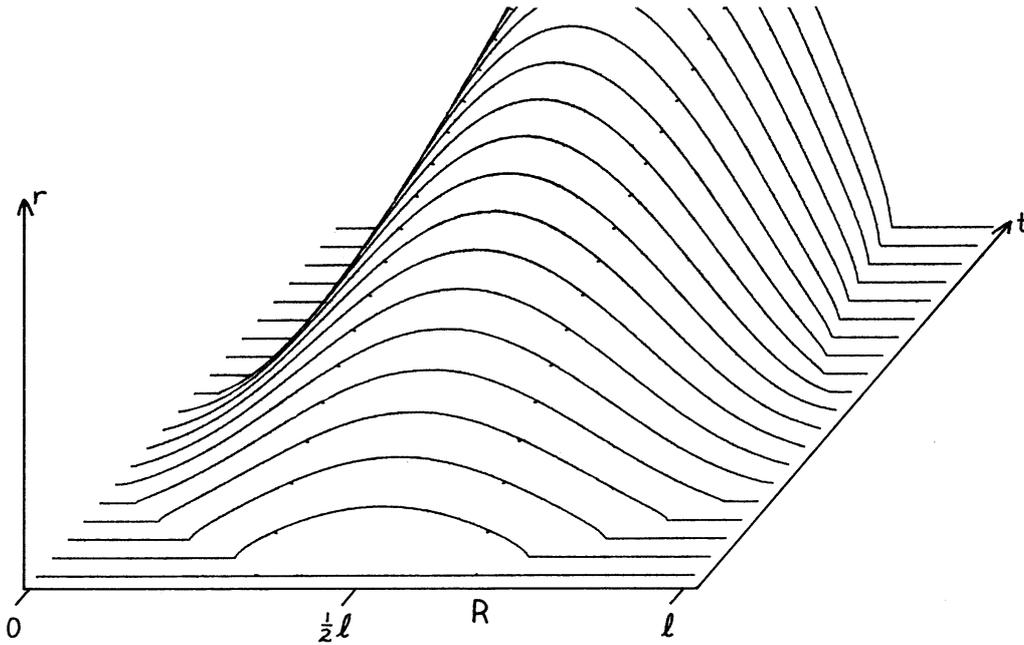


Fig. 4. The evolution of the function $r(R, t)$ for the metric functions given in §IIIc with $a_0 = 5$, $f_0 = 1$, $F_0 = 1$, $b = 3$, and $\ell = 1$. Successive curves, receding into the page, are the functions $r(R)$ at successively later times. Before the big bang and after recollapse, r has been set to zero. The divisions between the central hyperbolic region and the two elliptic regions are marked along each curve.

IV Discussion

The considerations given here were motivated by a recent paper by Zel'dovich and Grishchuk (1984). However, we have arrived at different conclusions.

First, the initial condition that they have chosen at time t_0 , which results in their equation (5), we consider to be restrictive. As they say, wherever $r'|_{t_0} = 0$ (R_0 , say), this condition requires

$$f(R_0) = -1, \quad (4.1)$$

which will not be true in general. Since the function $f(R)$ obeys $f \geq -1$, equation (4.1) also implies

$$f'(R_0) = 0. \quad (4.2)$$

Zel'dovich and Grishchuk also specify that the density is everywhere finite at t_0 , so equation (4.1) further implies

$$F'(R_0) = 0 \quad (4.3)$$

at R_0 . Then, by putting equations (4.2) and (4.3) in equation (2.6), we see that

$$a'(R_0) = 0 \quad (4.4)$$

is also required at R_0 . While conditions (4.2), (4.3), and (4.4) are all necessary for a regular extremum in r , (4.1) is not. Equations (4.2)-(4.4) also ensure that $r'(R_0, t) = 0$ holds at all times, and (4.1) ensures that recollapse occurs at R_0 , which is the result they obtain in their paper. In contrast, our example (§IIIc) explicitly demonstrates that a closed Tolman model, containing a hyperbolic region sandwiched between two elliptic regions, need not recollapse. Since no shell crossings form, the hyperbolic region remains and expands indefinitely. This violates their hypothesis that a closed hybrid model inevitably develops shell crossings which lead to the eventual recollapse of the model. The conditions derived in §II further show that models such as §IIIc are not hard to construct.

We have found that the conditions which ensure that shell crossings will not occur in Tolman metrics are quite easy to satisfy, and still allow a large range of physically interesting cases, so shell crossings certainly are not generic to the Tolman model. If one also requires a regular origin at $R = 0$ (or anywhere else), then the choice of arbitrary functions near the origin is further limited.

Previously (Hellaby and Lake 1984), we concluded that physically reasonable Tolman models must have a simultaneous bang, i.e., $a' = 0$ everywhere. This result is entirely compatible with the above conditions.

We agree with Zel'dovich and Grishchuk's speculation about what must physically happen when a shell crossing occurs — that separate particles no longer occupy separate points in space, and so there are three particle velocities at each point, but without a technique for dealing with such a process, one cannot say whether the subsequent spacetime expands or recollapses. However, we find that models which contain both elliptic and hyperbolic regions can be free of shell crossings, though they do contain surface layers of the kind commonly used in models of inhomogeneities. That some parts eventually collapse, while other parts continue to expand, presents no real problem as regards cosmic censorship, since the recollapse surface is spacelike.

We have also found that there is no necessary connection between the global geometry of a model (whether it is open or closed) and its time evolution. Of course, the local geometry is still related to the time evolution in the familiar manner, so the eventual fate of our part of the universe may still be determined by measuring the variation of the Hubble constant, H , with distance.

To obtain a model without surface layers or shell crossings, however, the condition $f(R_m) = -1$ is required, so that the surface mass is zero. It then follows from Table 1 that models which include a hyperbolic or parabolic section cannot be closed.

Acknowledgements

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Appendix A: General Properties of Shell Crossings

We show here that the properties of shell crossings mentioned in I apply to much more general metrics than Tolman's, and are not at all related to the equation of state.

Consider the general, spherically symmetric metric

$$ds^2 = -B^2 dt^2 + A^2 dR^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (\text{A1})$$

where A , B , and r are functions of t and R . The loci of points such that

$$A = 0, \quad r \neq 0, \quad B \neq 0, \quad (\text{A2})$$

are spherical three-surfaces and include regular maxima and shell crossings, as well as more singular surfaces. In general, all nonzero components of the Einstein tensor diverge at $A = 0$, but if they do not, we have a regular point such as the regular maxima.

It can be seen that the surface is timelike, by setting condition (A2) in (A1). The normal, n_α , to the surface, $A = \text{const}$, is

$$n_\alpha \propto (\dot{A}, A', 0, 0), \quad (\text{A3})$$

or, writing the locus of the surface as $t = b(r)$, then

$$n_\alpha \propto (1, -b', 0, 0), \quad (\text{A4})$$

so that

$$A' = -b' \dot{A}, \quad (\text{A5})$$

and $n_\alpha n^\alpha > 0$ gives $b'^2 > A^2/B^2$, which implies b' may have any value on $A = 0$, except zero.

The procedure for finding the redshift from an $A = 0$ surface is so close to that used in the Appendix of a previous paper (Hellaby and Lake 1984) that only the major steps are needed here. (Equation numbers from that paper will be preceded by "HL.") It is assumed that, near $A = 0$, the geodesic tangent vectors, k^α (null and timelike), may be expanded as a series in powers of A , i.e.,

$$k^t = \sum_{i=1}^{\infty} C_i(R) A^{\gamma_i}, \quad \text{and} \quad k^R = \sum_{i=1}^{\infty} D_i(R) A^{\delta_i}. \quad (\text{A6})$$

The geodesic equation,

$$k^\alpha \nabla_\alpha k^\beta = 0, \quad (\text{A7})$$

and the timelike or null condition,

$$k^\alpha k_\alpha = \epsilon , \quad (\text{A8})$$

are then solved to lowest order in powers of A . Thanks to spherical symmetry, we may choose to set $\theta = \pi/2$ for the motion, and we find

$$k^\theta = 0 , \quad \text{and} \quad k^\phi = \frac{h}{r^2} , \quad (\text{A9})$$

where h is an effective impact parameter. From equations (A1), and (A7)-(A9), the equivalent of equation (HL 3.13) is

$$4B^2 \left(A^2 k^{R^2} + \frac{h^2}{r^2} - \epsilon \right) [\partial_t(A^2 k^R)]^2 = \left\{ \partial_R \left[B^2 \left(A^2 k^{R^2} + \frac{h^2}{r^2} - \epsilon \right) \right] \right\}^2 . \quad (\text{A10})$$

Substituting from equations (A6) and (A5), we arrive at the equivalent of equation (HL A10) (the subscripts "1" on D and δ have been dropped),

$$\begin{aligned} & \left(D^2 A^{2\delta+2} + \frac{h^2}{r^2} - \epsilon \right) [\dot{D}A^{\delta+2} + (\delta+2)D\dot{A}A^{\delta+1}]^2 \\ & = \left\{ B' \left(D^2 A^{2\delta+2} + \frac{h^2}{r^2} - \epsilon \right) + B \left[DD'A^{2\delta+2} - (\delta+1)D^2 b' A^{2\delta+1} - \frac{h^2 r'}{r^3} \right] \right\}^2 . \end{aligned} \quad (\text{A11})$$

The only value of δ which gives a solution of equation (A11) which is real and which does not restrict the metric is $\delta = -1$, while D remains arbitrary; i.e.,

$$k^R = \frac{D}{A} , \quad k^t = \frac{1}{B} \sqrt{D^2 + \frac{h^2}{r^2} - \epsilon} . \quad (\text{A12})$$

Using a timelike geodesic for the emitter and a null geodesic for the light ray, we find the frequency shift to be

$$1 + z \propto \sqrt{\left(D_n^2 + \frac{h_n^2}{r^2} \right) \left(D_e^2 + \frac{h_e^2}{r^2} + 1 \right) - D_n D_e - \frac{h_n h_e \cos \psi}{r^2}} , \quad (\text{A13})$$

where D_n and h_n are constants of the null motion, D_e and h_e constants of the timelike motion, and ψ is the angle between the planes of the two trajectories. Clearly the frequency shift is always finite.

The third property is the surface energy density at $A = 0$. Following the procedure detailed recently by Lake (1984b), we consider a timelike surface, Σ , separating two manifolds, V^+ and V^- , both having metrics of the form (A1). The intrinsic metric of Σ is

$$ds^2 = -d\tau^2 + r^2(\tau)(d\theta^2 + \sin^2 \theta d\phi^2) . \quad (\text{A14})$$

The surface energy tensor, S_{ij} , is then defined by Lake's equation (2), and so the energy density, σ , is

$$4\pi\sigma = \frac{K_{\theta\theta}^- - K_{\theta\theta}^+}{r^2} = \frac{1}{r} \left| \kappa_- \sqrt{r^{*2} + \Gamma_-^2 - U_-^2} - \kappa_+ \sqrt{r^{*2} + \Gamma_+^2 - U_+^2} \right| \quad (\text{A15})$$

where K_{ij} is the extrinsic curvature of Σ , κ equals the sign of r' , the asterisk denotes $\partial/\partial\tau$, $\Gamma = r'/A$, $U = \dot{r}/B$, and all quantities are evaluated at Σ . If, on the plus side, say, we have

condition (A2), then σ is finite as long as Γ_+ is finite. Those surfaces for which σ does remain finite are shell crossings of the kind that appear in the Tolman metric (where $\Gamma^2 = 1 + f$; Lake 1984a), whereas those for which σ diverges are, in this sense, more singular.

The three properties of shell crossings given here are all derived from the metric form (A1), and no reference has been made to the equation of state.

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